$(43\text{-}44)_8$ Monday [2014-04-14] A1.3
15:00-18:00

Elastic energy

stress power density per unit current volume $\quad T \cdot \nabla v = T \cdot \dot{F}F^{-1}$

In an affine motion

$$T \cdot \dot{F}F^{-1} V_{\mathcal{R}} = T \cdot \dot{F}F^{-1} V_{\bar{\mathcal{R}}} \det F$$
$$= TF^{-T} \cdot \dot{F} \, V_{\bar{\mathcal{R}}} \det F$$
$$= \underbrace{S \cdot \dot{F}}_{} \, V_{\bar{\mathcal{R}}}$$

power density per unit reference volume

Hyperelastic material: there exists $\varphi$ such that

$$\hat{T}(F) \cdot \dot{F}F^{-1} V_{\mathcal{R}} = \frac{d}{dt} \varphi(F) \, V_{\bar{\mathcal{R}}}$$

in any (affine) motion

equivalently $\qquad \hat{S}(F) \cdot \dot{F} = \dfrac{d}{dt} \varphi(F)$

elastic energy per unit reference volume

Objectivity: elastic energy invariance under superposed rigid motion: $\varphi(F) = \varphi(QF)$

$Q = R^T \Rightarrow \qquad \varphi(F) = \varphi(R^T R U) = \varphi(U)$

necessary condition

 

Material symmetry

$$\varphi(F) = \varphi(FQ) \quad \Rightarrow Q \text{ belongs to the symmetry group}$$

Isotropic material

$$\forall Q \qquad \varphi(F) = \varphi(FQ) = \varphi(RUQ) = \varphi(\underbrace{RQ}_{\tilde{R}}\underbrace{Q^T UQ}_{\tilde{U}})$$

objectivity

$$\varphi(U) = \varphi(Q^T U Q)$$

isotropic energy function

Spectral decomposition of the stretch

$$U = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \qquad P_i = u_i \otimes u_i$$

unit eigenvector

$$\forall Q \qquad Q^T U Q = \lambda_1 Q^T P_1 Q + \lambda_2 Q^T P_2 Q + \lambda_3 Q^T P_3 Q$$

if $\quad u = P_i u$ (eigenvector of $U$)

then $\quad U u = \lambda_i u$

and $\quad (Q^T U Q)\, Q^T u = \lambda_i Q^T u$

$Q^T u$ eigenvector of $Q^T U Q$

isotropy $\quad \varphi(U) = \varphi(Q^T U Q)$

same eigenvalues
different eigenvectors

$\Rightarrow \quad \varphi(U)$ is independent of the eigenvectors of $U$

$(45\text{-}46)_8$ Tuesday [2014-04-15] A1.3

9:00 - 11:00

Isotropic elastic energy

$$\varphi(F) = \varphi(U) = \tilde{\varphi}(\lambda_1, \lambda_2, \lambda_3)$$

$$\varphi(U) = \breve{\varphi}(U^2) = \breve{\varphi}(C) = \tilde{\varphi}(\lambda_1^2, \lambda_2^2, \lambda_3^2)$$

$$= \hat{\varphi}(\iota_1, \iota_2, \iota_3)$$

principal invariants of C [iota]

coefficients of the characteristic polynomial

$$\det(C - \eta I) = \eta^3 - \iota_1 \eta^2 + \iota_2 \eta - \iota_3$$

$$\iota_1 = \operatorname{tr} C$$

$$\iota_2 = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right)$$

$$\iota_3 = \det C$$

$$\frac{d}{dt}\varphi(F) = \frac{d}{dt}\hat{\varphi}(\iota_1, \iota_2, \iota_3)$$

$$= \hat{\varphi}_{,1}\frac{d\iota_1}{dt} + \hat{\varphi}_{,2}\frac{d\iota_2}{dt} + \hat{\varphi}_{,3}\frac{d\iota_3}{dt}$$

$$\frac{d}{dt}\iota_1 = \frac{d}{dt}(F\cdot F) = 2F\cdot\dot F \qquad \iota_1 = \operatorname{tr}(F^TF) = F\cdot F$$

$$\frac{d}{dt}\iota_2 = 2\iota_1 F\cdot\dot F - \frac{1}{2}\frac{d}{dt}(C\cdot C)$$

$$= 2\iota_1 F\cdot\dot F - C\cdot\dot C = 2\iota_1 F\cdot\dot F - 2FC\cdot\dot F$$

$$C\cdot\dot C = C\cdot(\dot F^T F + F^T\dot F) = C\cdot((F^T\dot F)^T + F^T\dot F)$$

$$= 2C\cdot \operatorname{sym} F^T\dot F = 2C\cdot F^T\dot F = 2FC\cdot\dot F$$

$$\frac{d}{dt}\iota_3 = \frac{d}{dt}\det C = \frac{d}{dt}(\det F)^2 = 2\det F\frac{d}{dt}\det F$$

$$= 2\det F \ \det F \operatorname{tr}(\dot F F^{-1}) = 2\iota_3\operatorname{tr}(\dot F F^{-1})$$

$$= 2\iota_3\dot F\cdot F^{-T} = 2\iota_3 F^{-T}\cdot\dot F$$

$$= 2\iota_3 FC^{-1}\cdot\dot F$$

isotropic elastic energy

$$\frac{d}{dt}\varphi(F) = 2F\left(\left(\frac{\partial\hat\varphi}{\partial\iota_1} + \frac{\partial\hat\varphi}{\partial\iota_2}\iota_1\right)\underline{I} - \frac{\partial\hat\varphi}{\partial\iota_2}C + \frac{\partial\hat\varphi}{\partial\iota_3}\iota_3 C^{-1}\right)\cdot\dot F$$

$$\hat S(F)\cdot\dot F = \frac{d}{dt}\varphi(F)$$

$$\hat{S}(F) \cdot \dot{F} = \left(\hat{T}(F) F^{-T} \det F\right) \cdot \dot{F} = \hat{T}(F) \cdot \dot{F} F^{-1} \det F$$

$$\hat{T}(F) = \hat{S}(F) F^{T} \frac{1}{\det F}$$

$$\hat{T}(F) = \frac{2}{\sqrt{\iota_3}}\left(\left(\frac{\partial\hat{\varphi}}{\partial\iota_1} + \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_1\right)B - \frac{\partial\hat{\varphi}}{\partial\iota_2}B^2 + \frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 I\right)$$

$B = FF^T$ left Cauchy-Green tensor

$$F C F^T = F F^T F F^T = B^2$$

$$F C^{-1} F^T = F(F^{-1} F^{-T}) F^T = I$$

Second Piola stress tensor

$$S \cdot \dot{F} = \frac{1}{2} S_{II} \cdot \dot{C}$$

$C = C^T \Rightarrow S_{II}$ is a symmetric tensor

$$\frac{1}{2} S_{II} \cdot \dot{C} = \frac{1}{2} S_{II} \cdot (\dot{F}^T F + F^T \dot{F}) = S_{II} \cdot \frac{1}{2}\left((F^T\dot{F})^T + (F^T\dot{F})\right)$$

$$= S_{II} \cdot \operatorname{sym} F^T \dot{F} = S_{II} \cdot F^T \dot{F} = F S_{II} \cdot \dot{F}$$

$$\Rightarrow \quad S = F S_{II} \Rightarrow S_{II} = F^{-1} S = F^{-1} T F^{-T} \det I$$

 

$(47-48)_8$ Wednesday [2014-04-16] A1.3
9:00-11:00

Force balance principle

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \quad \forall v$$

Power expended

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dV = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

by using the Piola stress

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\mathcal{R}} T \cdot \dot{F} F^{-1} dV = \int_{\bar{\mathcal{R}}} S \cdot \dot{F} \, dV$$

Energy balance principle (dissipation inequality)

$$S \cdot \dot{F} - \frac{d}{dt} \varphi(F) \geq 0$$

$$S \cdot \dot{F} - \hat{S}(F) \cdot \dot{F} \geq 0$$

$$\overbrace{\left(S - \hat{S}(F)\right)}^{S^+} \cdot \dot{F} \geq 0 \qquad S^+ \cdot \dot{F} \geq 0$$

$$\underbrace{\left(T - \hat{T}(F)\right)}_{T^+} \cdot \dot{F} F^{-1} \geq 0 \qquad T^+ \cdot \dot{F} F^{-1} \geq 0$$

$$T = \hat{T}(F) + T^+$$

Possible choice (viscous stress)

$$T^+ := 2\mu \, \mathrm{sym} \, \dot{F} F^{-1}$$

$$\left(\mathrm{sym} \, \dot{F} F^{-1}\right) \cdot \dot{F} F^{-1} \geq 0$$

viscosity $\qquad \mu \geq 0 \Leftrightarrow T^+ \cdot \dot{F} F^{-1} \geq 0$

 

Incompressible materials

$$\det F = 1 \quad \text{isochoric deformation (motion)}$$

$$\frac{d}{dt}\det F = (\det F)\,\mathrm{tr}(\dot F F^{-1}) = 0$$

$$\Rightarrow \quad \mathrm{tr}(\dot F F^{-1}) = 0 \iff \mathrm{tr}\,\nabla v = 0 \iff \mathrm{div}\, v = 0$$

$$aI \cdot \nabla v = a\,\mathrm{tr}\,\nabla v = 0 \qquad \forall \text{ spherical tensor } aI$$

$$\mathrm{sph}\,T := \frac{1}{3}(\mathrm{tr}\,T)I \Rightarrow (\mathrm{sph}\,T)\cdot\nabla v = 0$$

$$\mathrm{dev}\,T := T - \mathrm{sph}\,T \Rightarrow \mathrm{tr}(\mathrm{dev}\,T) = \mathrm{tr}\,T - \frac{1}{3}(\mathrm{tr}\,T)3 = 0$$

$$T = \underbrace{\mathrm{sph}\,T}_{\text{spherical part}} + \underbrace{\mathrm{dev}\,T}_{\text{deviatoric part}}$$

$$\underbrace{\mathrm{tr}\,\nabla v = 0}_{\text{isochoric velocity field}} \Rightarrow T\cdot\nabla v = (\mathrm{sph}\,T + \mathrm{dev}\,T)\cdot\nabla v = \mathrm{dev}\,T\cdot\nabla v$$

$$\det F = 1 \Rightarrow \hat S(F)\cdot\dot F = \hat T(F)\cdot\dot F F^{-1} = \mathrm{dev}\,\hat T(F)\cdot\dot F F^{-1}$$

elastic stress $\quad \mathrm{dev}\,\hat T(F)\cdot\dot F F^{-1} = \dfrac{d}{dt}\varphi(F)$

reactive stress $\quad \mathrm{sph}\,T = -pI$

internal pressure

only the deviatoric part of the stress is determined by the energy because of the incompressibility

$$\Rightarrow \quad \underbrace{\left(\operatorname{dev} T - \hat{T}(F)\right)}_{T^{+} \text{ is a deviatoric tensor}} \cdot \dot{F} F^{-1} \geq 0$$

$$T = \operatorname{dev} T + \operatorname{sph} T = \hat{T}(F) + T^{+} - pI$$

$$T = \underbrace{\hat{T}(F)}_{\substack{\text{elastic} \\ \text{deviatoric}}} - \underbrace{pI}_{\substack{\text{reactive} \\ \text{spherical}}} + \underbrace{T^{+}}_{\substack{\text{dissipative} \\ \text{deviatoric}}}$$