$(57\text{-}58)_{11}$ Monday [2014-05-05] A1.3

16:00 – 18:00

A standard problem in elasticity

uniform traction

cubic reference shape

Let us consider "cylindrical deformations" only:

affine deformations with

$$F e_1 = \lambda_1 e_1 \qquad F e_2 = \lambda_2 e_2 \qquad F e_3 = \lambda_3 e_3 \qquad [F] = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

moment tensor

$$\begin{aligned} M &= \bar{\ell}\, F e_1 \otimes (p e_1) A_{\mathcal{F}_1} \\ &= \bar{\ell}\, \lambda_1\, e_1 \otimes e_1\, (p A_{\mathcal{F}_1}) \\ &= \ell_1\, e_1 \otimes e_1\, (p\, \ell_2 \ell_3) \\ &= \underbrace{\ell_1 \ell_2 \ell_3}_{\text{current volume } V_{\mathcal{R}}}\, p\, e_1 \otimes e_1 \end{aligned}$$

balance equations

$$f = 0$$

$$\operatorname{skw} M = 0$$

$$\frac{M}{V_{\mathcal{R}}} = T$$

from balance $\quad T = p\, e_1 \otimes e_1$

material characterization

$$T = \hat{T}(F) - p I + T^+$$

(incompressibility) (dissipation)

Let us consider a compressible material with no dissipation:

$$T = \hat{T}(F)$$

with

$$[\hat{T}(F)] = \begin{pmatrix} \hat{\sigma}_{11}(F) & & \\ & \hat{\sigma}_{22}(F) & \\ & & \hat{\sigma}_{33}(F) \end{pmatrix}$$

$$[\dot{F} F^{-1}] = \begin{pmatrix} \frac{\dot{\lambda}_1}{\lambda_1} & & \\ & \frac{\dot{\lambda}_2}{\lambda_2} & \\ & & \frac{\dot{\lambda}_3}{\lambda_3} \end{pmatrix}$$

balance

$$\begin{pmatrix} \sigma_{11} & & \\ & \sigma_{22} & \\ & & \sigma_{33} \end{pmatrix} = \begin{pmatrix} p & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$T = p \, e_1 \otimes e_1$$

A simple choice for $\hat{T}(F)$ is

$$\hat{\sigma}_{11}(F) = a_1(\lambda_1 - 1)$$

$$\hat{\sigma}_{22}(F) = a_2(\lambda_2 - 1)$$

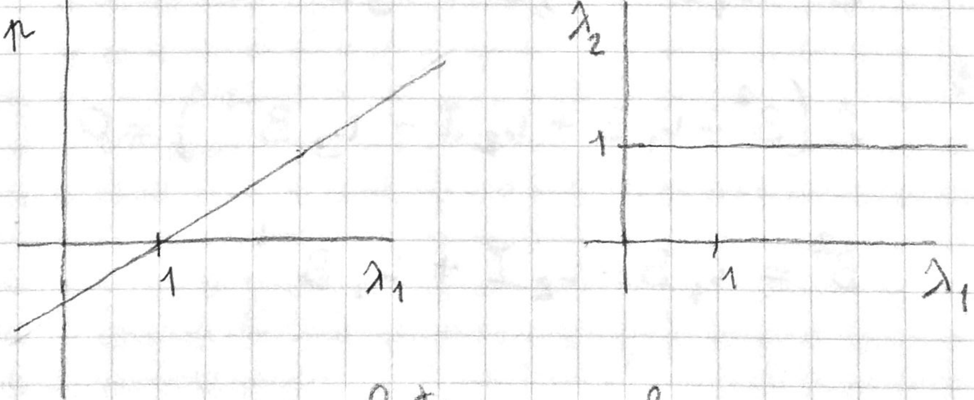
$$\hat{\sigma}_{33}(F) = a_3(\lambda_3 - 1)$$

Balance and material response

$$p = a_1(\lambda_1 - 1)$$

$$0 = a_2(\lambda_2 - 1)$$

$$0 = a_3(\lambda_3 - 1)$$



solution graphs

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9:00-11:00

The solution we derived from the assumed material response is unsatisfactory because it allows λ_1 to be negative; further λ_2 and λ_3 are always equal to 1.

Let us consider the response function for a hyperelastic material $\rightarrow (45\text{-}46)_8$

$$\hat{T}(F) = \frac{2}{\sqrt{\iota_3}}\left(\left(\frac{\partial\hat{\varphi}}{\partial\iota_1} + \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_1\right)B - \frac{\partial\hat{\varphi}}{\partial\iota_2}B^2 + \frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 I\right)$$

with $B = FF^T$ (left Cauchy-Green tensor)

We can get a new expression by using the Caley-Hamilton theorem stating that any tensor satisfies its characteristic equation:

$$B^3 - \iota_1 B^2 + \iota_2 B - \iota_3 I = 0$$

We get

$$B\left(B^2 - \iota_1 B + \iota_2 I - \iota_3 B^{-1}\right) = 0$$

$$B^2 = \iota_1 B - \iota_2 I + \iota_3 B^{-1}$$

$$\hat{T}(F) = \frac{2}{\sqrt{\iota_3}}\left(\frac{\partial\hat{\varphi}}{\partial\iota_1}B + \left(\frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 + \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_2\right)I - \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_3 B^{-1}\right)$$

If
$$[F] = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

then
$$[B] = \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix}$$

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9:00 - 11:00

let us consider an incompressible material.

The motion will be isochoric: any deformation is characterized by $\det F = 1$.

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

A standard strain energy for so called rubber-like material is

$$\varphi_H(F) = c_1(\iota_1 - 3) \qquad \text{neo-Hookean}$$

or

$$\varphi_{MR}(F) = c_1(\iota_1 - 3) + c_2(\iota_2 - 3)$$

Mooney-Rivlin

For the assumed deformation gradient we have

$$[C] = [F^T F] = \begin{pmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{pmatrix}$$

$$\iota_1 = \operatorname{tr} C = \lambda^2 + \frac{2}{\lambda}$$

$$[C^2] = \begin{pmatrix} \lambda^4 & & \\ & \frac{1}{\lambda^2} & \\ & & \frac{1}{\lambda^2} \end{pmatrix}$$

$$\iota_2 = \frac{1}{2}\left((\mathrm{tr}\, C)^2 - \mathrm{tr}\, C^2\right) = \frac{1}{2}\left(\lambda^4 + 4\lambda + \frac{4}{\lambda^2} - \lambda^4 - \frac{2}{\lambda^2}\right) = 2\lambda + \frac{1}{\lambda^2}$$

$$\iota_3 = 1$$

$$[\dot{F}] = \begin{pmatrix} \dot{\lambda} & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda} \end{pmatrix}$$

$$[\dot{F}F^{-1}] = \begin{pmatrix} \frac{\dot{\lambda}}{\lambda} & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\,\lambda^{-\frac{1}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\,\lambda^{-\frac{1}{2}} \end{pmatrix}$$

$$[\dot{F}F^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix}\frac{\dot{\lambda}}{\lambda}$$

note $\mathrm{tr}\, \dot{F}F^{-1} = 0$