[2016-05-23]

Linear elasticity (small deformations)

$$F(\varepsilon) = I + E(\varepsilon) + \textcircled{H}(\varepsilon) + o(\varepsilon)$$

$$F(\varepsilon) = R(\varepsilon)\, U(\varepsilon)$$

$$R(\varepsilon) = R(0) + \frac{d}{d\varepsilon} R\Big|_{\varepsilon=0} \varepsilon = I + \textcircled{H} + o(\varepsilon)$$

$$U(\varepsilon) = U(0) + \frac{d}{d\varepsilon} U\Big|_{\varepsilon=0} \varepsilon = I + E + o(\varepsilon)$$

$$R(\varepsilon)^T R(\varepsilon) = I \Rightarrow (I + \textcircled{H}^T)(I + \textcircled{H}) = I$$

$$I + \textcircled{H} + \textcircled{H}^T + o(\varepsilon) = I$$

$$\Rightarrow \quad \textcircled{H} + \textcircled{H}^T = 0$$

$$U(\varepsilon) = U(\varepsilon)^T \Rightarrow I + E = I + E^T$$

$$\Rightarrow \quad E = E^T$$

$\textcircled{H}$ infinitesimal rotation

E infinitesimal strain

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$$F^{T}F = \left(I + E^{T} + \Theta^{T}\right)\left(I + E + \Theta\right)$$

$$= I + E + \Theta + E^{T} + \Theta^{T} = I + 2E + o(\varepsilon)$$

$$\iota_1 = 3 + 2\operatorname{tr}E + o(\varepsilon)$$

$$\det F = 1 + \left(\frac{d}{d\varepsilon}\det F(0)\right)\varepsilon + o(\varepsilon)$$

$$= 1 + \operatorname{tr}E + \operatorname{tr}\Theta + o(\varepsilon)$$

$$= 1 + \operatorname{tr}E + o(\varepsilon)$$

Isotropic elastic (compressible neo-Hookean) material

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$$\hat{T}(F) = 2k_{I}\, J^{-\frac{5}{3}}\left(B - \frac{1}{3}\iota_1 I\right) + 2k_V(J-1)I$$

$$B - \frac{1}{3}\iota_1 I = I + 2E - \frac{1}{3}(3 + 2\,\mathrm{tr}E)I + o(\varepsilon)$$

$$= I + 2E - I - \frac{2}{3}(\mathrm{tr}E)I + o(\varepsilon)$$

$$= 2\left(E - \frac{1}{3}(\mathrm{tr}E)I\right) + o(\varepsilon)$$

$$J^{-\frac{5}{3}} = 1 - \frac{5}{3}\left(\frac{d}{d\varepsilon}\det F(\varepsilon)\right)\varepsilon + o(\varepsilon) = 1 - \frac{5}{3}\,\mathrm{tr}E + o(\varepsilon)$$

$$J^{-\frac{5}{3}}\left(B - \frac{1}{3}\iota_1 I\right) = 2\left(E - \frac{1}{3}(\mathrm{tr}E)I\right) + o(\varepsilon)$$

$$\hat{T}(F) = 4k_{I}\left(E - \frac{1}{3}(\mathrm{tr}E)I\right) + 2k_V(\mathrm{tr}E)I$$

$$= \left(2k_V - \frac{4}{3}k_{I}\right)(\mathrm{tr}E)I + 4k_{I}E$$

$$= \underbrace{\left(\kappa - \frac{2}{3}\mu_L\right)}(\mathrm{tr}E)I + 2\mu_L E$$

$$= \lambda_L(\mathrm{tr}E)I + 2\mu_L E \qquad (\text{Lamé})$$

$$\kappa - \frac{2}{3}\mu_L = \lambda_L \;\Rightarrow\; \kappa = \lambda_L + \frac{2}{3}\mu_L = \frac{1}{3}(3\lambda_L + 2\mu_L)$$

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Linear elasticity

isotropic material

$$\widehat{T}(F) = \lambda_L (\text{tr}\, E) I + 2\mu_L E$$

λ_L, μ_L Lamé moduli

$\mu_L = 2k_I$ shear modulus

$\kappa = 2k_V$ bulk modulus

$\lambda_L = \kappa - \dfrac{2}{3}\mu_L$

From uniaxial deformation (linearized)

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Uniaxial deformation

$$[F] = J^{\frac{1}{3}} \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} = \begin{pmatrix} \lambda_a & & \\ & \lambda_r & \\ & & \lambda_r \end{pmatrix}$$

$$\begin{cases} \lambda_a = \lambda \, J^{\frac{1}{3}} \\ \lambda_r = \frac{1}{\sqrt{\lambda}} J^{\frac{1}{3}} \end{cases} \Rightarrow \begin{cases} \frac{\lambda_a}{\lambda_r} = \lambda \sqrt{\lambda} \Rightarrow \lambda = \left(\frac{\lambda_a}{\lambda_r}\right)^{\frac{2}{3}} \\ J = \lambda_a \lambda_r^2 \end{cases}$$

$$[\dots] \begin{cases} \dfrac{n}{k_I} = 2 J \left(\lambda^2 - \dfrac{1}{\lambda}\right) \\ \dfrac{n}{k_V} = 6 \left(J - 1\right) \end{cases}$$

$$\lambda_a = 1 + \varepsilon_a \quad , \quad \lambda_r = 1 + \varepsilon_r$$

$$J = 1 + \varepsilon_a + 2 \varepsilon_r \qquad \qquad \text{(small strains)}$$

$$\lambda = 1 + \frac{2}{3} \left(\varepsilon_a - \varepsilon_r\right)$$

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$$2J\left(\lambda^2 - \frac{1}{\lambda}\right) = 0 + 2(2+1)\frac{2}{3}\left(\varepsilon_a - \varepsilon_r\right)$$

$$= 4\left(\varepsilon_a - \varepsilon_r\right)$$

$$\begin{cases} \dfrac{\mu}{2k_I} = 2\left(\varepsilon_a - \varepsilon_r\right) \\[3em] \dfrac{\mu}{2k_V} = 3\left(2\varepsilon_r + \varepsilon_a\right) \end{cases}$$

$$\begin{cases} \dfrac{\mu}{\mu_L} = 2\left(\varepsilon_a - \varepsilon_r\right) \\[3em] \dfrac{\mu}{k} = 3\left(2\varepsilon_r + \varepsilon_a\right) \end{cases}$$

$$\begin{cases} \dfrac{\mu}{2\mu_L} = \varepsilon_a - \varepsilon_r \\[2.5em] \dfrac{\mu}{3k} = \varepsilon_a + 2\varepsilon_r \end{cases}$$

$$\mu\left(\frac{1}{\mu_L} + \frac{1}{3k}\right) = 3\varepsilon_a$$

$$\mu\left(-\frac{1}{2\mu_L} + \frac{1}{3k}\right) = 3\varepsilon_r$$

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$$k = \lambda_L + \frac{2}{3}\mu_L = \frac{1}{3}\left(3\lambda_L + 2\mu_L\right)$$

$$\mu = 3\,\frac{3\mu_L k}{3k + \mu_L}\,\varepsilon_a$$

$$= 3\,\frac{\mu_L\left(3\lambda_L + 2\mu_L\right)}{3\lambda_L + 2\mu_L + \mu_L}\,\varepsilon_a$$

$$= \underbrace{\frac{\mu_L\left(3\lambda_L + 2\mu_L\right)}{\lambda_L + \mu_L}}_{Y}\,\varepsilon_a$$
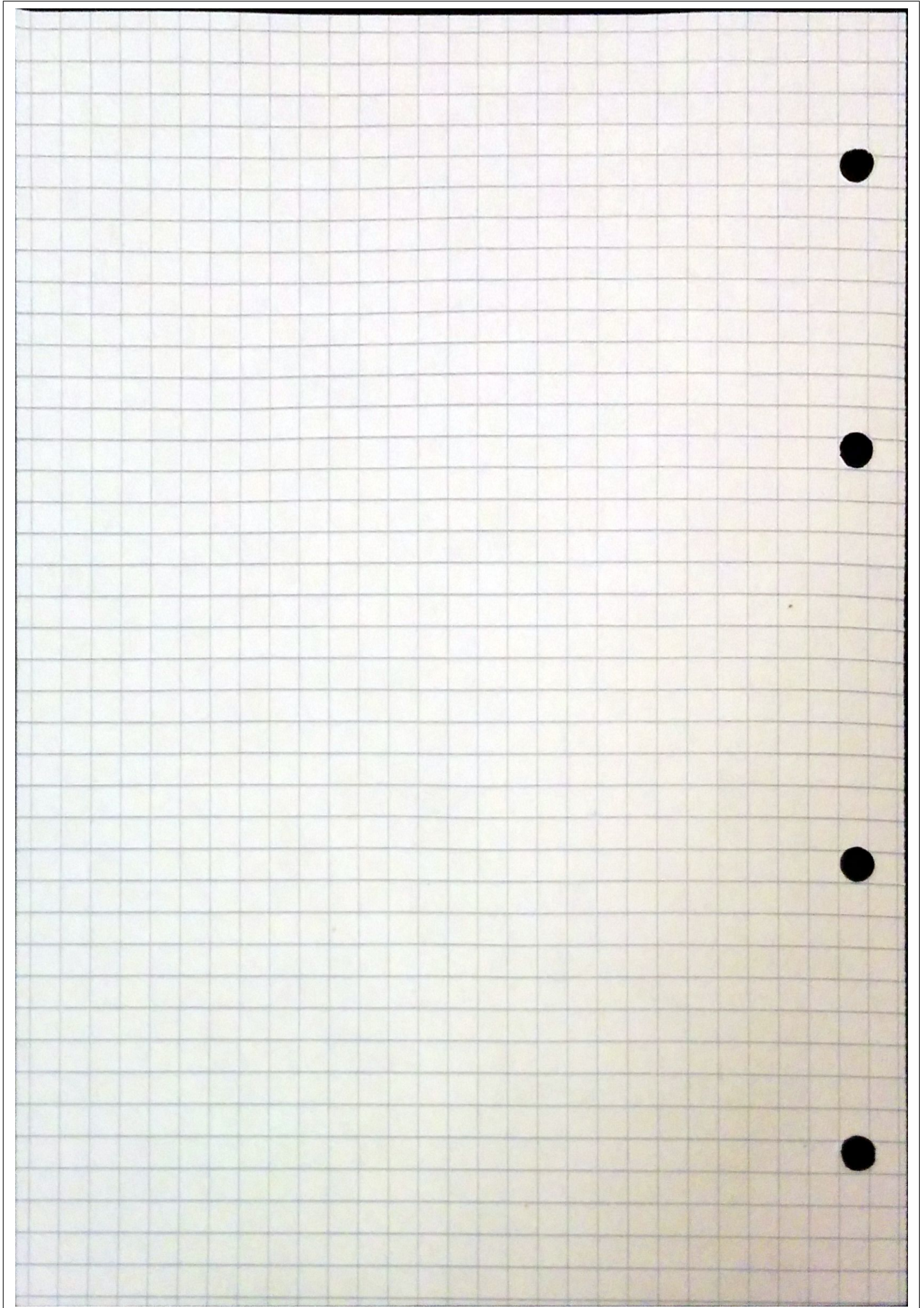
Young's modulus

$$\frac{\varepsilon_r}{\varepsilon_a} = \frac{-3k + 2\mu_L}{6\mu_L k}\,\frac{3\mu_L k}{3k + \mu_L}$$

$$= \frac{-3\lambda_L - 2\mu_L + 2\mu_L}{2\left(3\lambda_L + 2\mu_L + \mu_L\right)}$$

$$= -\underbrace{\frac{\lambda_L}{2\left(\lambda_L + \mu_L\right)}}_{}$$

Poisson's ratio

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