Rigid and affine motions

Contents

1 Motions

By *motion of a body* $\mathcal{B}$ we mean a regular function

$$\mathsf{p} : \mathcal{B} \times \mathbb{R} \to \mathcal{E} \tag{1}$$

such that for any t

$$\mathsf{p}(\cdot, t) : \mathcal{B} \to \mathcal{E} \tag{2}$$

is a placement and

$$\mathsf{p}_\mathsf{A}(\cdot) \equiv \mathsf{p}(\mathsf{A}, \cdot) : \mathbb{R} \to \mathcal{E} \tag{3}$$

is the motion of any body point A. Hence a motion is a one-parameter family of placements. A motion can also be described as a one-parameter family of deformations by defining

$$\boldsymbol{\phi} : \bar{\mathcal{R}} \times \mathbb{R} \to \mathcal{E} \tag{4}$$

transforming the position $\bar{\mathsf{p}}_\mathsf{A} \equiv \mathsf{p}(\mathsf{A}, t_0) \in \bar{\mathcal{R}}$ of each body point $\mathsf{A} \in \mathcal{B}$ at time t_0 into its position at time t

$$\mathsf{p}_\mathsf{A}(t) = \mathsf{p}(\mathsf{A}, t) = \boldsymbol{\phi}(\mathsf{p}(\mathsf{A}, t_0), t). \tag{5}$$

By $\bar{\mathcal{R}}$ we shall denote the shape of the body at a time t_0. We call the image in $\mathcal{E}$ of p_A the *trajectory* of the body point A. The *velocity* at time t of the point A is the vector

$$\dot{\mathsf{p}}_\mathsf{A}(t) := \lim_{\tau \to 0} \frac{1}{\tau}(\mathsf{p}_\mathsf{A}(t+\tau) - \mathsf{p}_\mathsf{A}(t)). \tag{6}$$

After setting a coordinate system by selecting an origin $\boldsymbol{o}$ and an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, a description of the motion can be given in terms of coordinates as follows

$$\mathsf{p}_\mathsf{A}(t) = \boldsymbol{o} + x_{1\mathsf{A}}(t)\mathbf{e}_1 + x_{2\mathsf{A}}(t)\mathbf{e}_2 + x_{3\mathsf{A}}(t)\mathbf{e}_3. \tag{7}$$

The expression for the velocity turns out to be

$$\begin{aligned} \dot{\mathsf{p}}_\mathsf{A}(t) &:= \lim_{\tau \to 0} \frac{1}{\tau}((x_{1\mathsf{A}}(t+\tau) - x_{1\mathsf{A}}(t))\mathbf{e}_1 + (x_{2\mathsf{A}}(t+\tau) - x_{2\mathsf{A}}(t))\mathbf{e}_2 + (x_{3\mathsf{A}}(t+\tau) - x_{3\mathsf{A}}(t))\mathbf{e}_3) \\ &= \frac{d}{dt}x_{1\mathsf{A}}(t)\mathbf{e}_1 + \frac{d}{dt}x_{2\mathsf{A}}(t)\mathbf{e}_2 + \frac{d}{dt}x_{3\mathsf{A}}(t)\mathbf{e}_3. \end{aligned} \tag{8}$$

By *test velocity field* we mean the collection of the body point velocities

$$\dot{\mathsf{p}}_\mathsf{A}(t), \dot{\mathsf{p}}_\mathsf{B}(t), \ldots \tag{9}$$

2 Rigid motions

A motion is *rigid* if, for any t_0, the deformations (4) are such that for any two body points A e B and at any time t

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_\mathsf{B}, t) = \boldsymbol{\phi}(\bar{\mathsf{p}}_\mathsf{A}, t) + \mathbf{R}(t)(\bar{\mathsf{p}}_\mathsf{B} - \bar{\mathsf{p}}_\mathsf{A}), \tag{10}$$

where $\mathbf{R}(t)$ is a rotation. Hence a *rigid motion* turns out to defined by the motion of any body point, say A, and by $\mathbf{R}$. The corresponding expression for the velocity is

$$\dot{\boldsymbol{\phi}}(\bar{\mathsf{p}}_\mathsf{B}, t) = \dot{\boldsymbol{\phi}}(\bar{\mathsf{p}}_\mathsf{A}, t) + \dot{\mathbf{R}}(t)(\bar{\mathsf{p}}_\mathsf{B} - \bar{\mathsf{p}}_\mathsf{A}). \tag{11}$$

Replacing (10) with

$$\mathsf{p}_B(t) = \mathsf{p}_A(t) + \mathbf{R}(t)(\bar{\mathsf{p}}_B - \bar{\mathsf{p}}_A), \tag{12}$$

the velocity (11) becomes

$$\dot{\mathsf{p}}_B(t) = \dot{\mathsf{p}}_A(t) + \dot{\mathbf{R}}(t)(\bar{\mathsf{p}}_B - \bar{\mathsf{p}}_A). \tag{13}$$

Note that from (12) we get

$$\bar{\mathsf{p}}_B - \bar{\mathsf{p}}_A = \mathbf{R}(t)^\mathsf{T}(\mathsf{p}_B(t) - \mathsf{p}_A(t)). \tag{14}$$

By substituting this expression into (13) we obtain

$$\dot{\mathsf{p}}_B(t) = \dot{\mathsf{p}}_A(t) + \dot{\mathbf{R}}(t)\mathbf{R}(t)^\mathsf{T}(\mathsf{p}_B(t) - \mathsf{p}_A(t)). \tag{15}$$

Setting

$$\mathbf{W}(t) := \dot{\mathbf{R}}(t)\mathbf{R}(t)^\mathsf{T}, \tag{16}$$

allows to rewrite expression (15) as

$$\dot{\mathsf{p}}_B(t) = \dot{\mathsf{p}}_A(t) + \mathbf{W}(t)(\mathsf{p}_B(t) - \mathsf{p}_A(t)). \tag{17}$$

The tensor $\mathbf{W}(t)$, called the *spin tensor*, turns out to be skew symmetric. To show this let us note that if $\mathbf{R}(t)$ is a rotation then $\mathbf{R}(t)^{-1} = \mathbf{R}(t)^\mathsf{T}$ is a rotation as well. Hence

$$\mathbf{R}(t)\mathbf{R}(t)^\mathsf{T} = \mathbf{I}. \tag{18}$$

Differentiating with respect to t we get

$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^\mathsf{T} + \mathbf{R}(t)\dot{\mathbf{R}}(t)^\mathsf{T} = \mathbf{O} \quad \Rightarrow \quad \mathbf{W}(t) + \mathbf{W}(t)^\mathsf{T} = \mathbf{O} \quad \Rightarrow \quad \mathbf{W}(t)^\mathsf{T} = -\mathbf{W}(t). \tag{19}$$

Further, setting

$$\mathbf{d}(t) := \mathsf{p}_B(t) - \mathsf{p}_A(t), \tag{20}$$

from (17) we get

$$\dot{\mathbf{d}}(t) = \mathbf{W}(t)\mathbf{d}(t). \tag{21}$$

Note that by the skew symmetry of the spin

$$\dot{\mathbf{d}}(t) \cdot \mathbf{d}(t) = \mathbf{W}(t)\mathbf{d}(t) \cdot \mathbf{d}(t) = \mathbf{d}(t) \cdot \mathbf{W}(t)^\mathsf{T}\mathbf{d}(t) = -\mathbf{d}(t) \cdot \mathbf{W}(t)\mathbf{d}(t). \tag{22}$$

Hence

$$\dot{\mathbf{d}}(t) \cdot \mathbf{d}(t) = 0, \tag{23}$$

$$\mathbf{W}(t)\mathbf{d}(t) \cdot \mathbf{d}(t) = 0. \tag{24}$$

By the first property the velocity difference

$$\dot{\mathbf{d}}(t) = \dot{\mathsf{p}}_B(t) - \dot{\mathsf{p}}_A(t) \tag{25}$$

is either zero or orthogonal to the difference vector (20). By the second property $\mathbf{W}(t)$ transforms any vector $\mathbf{d}(t)$ into a vector orthogonal to $\mathbf{d}(t)$. A *rigid velocity field* is a velocity field satisfying (17), with $\mathbf{W}(t)$ skew symmetric.

3 Spin axis

A skew symmetric tensor $\mathbf{W}(t)$, as an endomorphism of a real vector space of dimension three, has a null eigenvalue. In fact, since the characteristic polynomial is of order three there exists at least one real eigenvalue λ. Denoting by $\mathbf{a}_o(t)$ a unit eigenvector corresponding to λ it is

$$\mathbf{W}(t)\mathbf{a}_o(t) = \lambda\, \mathbf{a}_o(t). \tag{26}$$

By (24)

$$\lambda = \mathbf{W}(t)\mathbf{a}_o(t) \cdot \mathbf{a}_o(t) = 0. \tag{27}$$

Hence

$$\mathbf{W}(t)\mathbf{a}_o(t) = \mathbf{o}. \tag{28}$$

Let us consider, at a time t, a line passing through $\mathsf{p}_\mathsf{A}(t)$

$$c_o(h,t) = \mathsf{p}_\mathsf{A}(t) + h\, \mathbf{a}_o(t). \tag{29}$$

All body points occupying such positions have the same velocity since by (28),

$$\dot{c}_o(h,t) = \dot{\mathsf{p}}_\mathsf{A}(t) + \mathbf{W}(t)(c_o(h,t) - \mathsf{p}_\mathsf{A}(t)) = \dot{\mathsf{p}}_\mathsf{A}(t) + h\mathbf{W}(t)\mathbf{a}_o(t) = \dot{\mathsf{p}}_\mathsf{A}(t). \tag{30}$$

This property holds for any line parallel to $\mathbf{a}_o(t)$. Hence for each line parallel to $\mathbf{a}_o(t)$ there is a common velocity, possibly different from line to line. Let us consider now any two body points A e B and the difference vector (20). Note that the velocity difference (25) is such that, by (28),

$$\dot{\mathbf{d}}(t) \cdot \mathbf{a}_o(t) = \mathbf{W}(t)\mathbf{d}(t) \cdot \mathbf{a}_o(t) = \mathbf{d}(t) \cdot \mathbf{W}(t)^\mathsf{T}\mathbf{a}_o(t) = -\mathbf{d}(t) \cdot \mathbf{W}(t)\mathbf{a}_o(t) = 0. \tag{31}$$

Hence the velocity difference is either zero or orthogonal to $\mathbf{d}(t)$, by (23), and to $\mathbf{a}_o(t)$ as well. Property (31) can be also given a different interpretation. If it is put in the form

$$(\dot{\mathsf{p}}_\mathsf{B}(t) - \dot{\mathsf{p}}_\mathsf{A}(t)) \cdot \mathbf{a}_o(t) = 0, \tag{32}$$

it implies that

$$\dot{\mathsf{p}}_\mathsf{B}(t) \cdot \mathbf{a}_o(t) = \dot{\mathsf{p}}_\mathsf{A}(t) \cdot \mathbf{a}_o(t), \tag{33}$$

Thus the orthogonal projection of the velocity on $\mathbf{a}_o(t)$ turns out to be the same for all body points. Hence the velocity of each body point can be decomposed into the sum of a velocity $\mathbf{v}_o(t)$ parallel to the axis $\mathbf{a}_o(t)$, which is unique for the whole body, and a velocity orthogonal to $\mathbf{a}_o(t)$.

Let us consider a straight line passing through $\mathsf{p}_\mathsf{A}(t)$ and lying on a plane orthogonal to $\mathbf{a}_o(t)$

$$c(h,t) = \mathsf{p}_\mathsf{A}(t) + h\mathbf{d}(t), \qquad \mathbf{d}(t) \cdot \mathbf{a}_o(t) = 0. \tag{34}$$

Velocities along this line can be expressed as

$$\dot{c}(h,t) = \dot{\mathsf{p}}_\mathsf{A}(t) + h\mathbf{W}(t)\mathbf{d}(t) = \mathbf{v}_o(t) + \mathbf{v}_\mathsf{A}^\perp(t) + h\mathbf{W}(t)\mathbf{d}(t), \tag{35}$$

with

$$\mathbf{v}_o(t) := (\dot{\mathsf{p}}_\mathsf{A}(t) \cdot \mathbf{a}_o(t))\mathbf{a}_o(t), \tag{36}$$

$$\mathbf{v}_\mathsf{A}^\perp(t) := (\dot{\mathsf{p}}_\mathsf{A}(t) - \mathbf{v}_o(t)), \tag{37}$$

where $\mathbf{v}_\mathsf{A}^\perp(t)$, like $\mathbf{d}(t)$, is a vector orthogonal to $\mathbf{a}_o(t)$.

If we choose $\mathbf{d}(t)$ orthogonal also to $\mathbf{v}_\mathsf{A}^\perp(t)$ then $\mathbf{W}(t)\mathbf{d}(t)$, which is orthogonal both to $\mathbf{d}(t)$ and $\mathbf{a}_o(t)$ by (24) and (31), turns out to be parallel to $\mathbf{v}_\mathsf{A}^\perp(t)$. Then there exists a unique value for h such that

$$\mathbf{v}_\mathsf{A}^\perp(t) + h\mathbf{W}(t)\mathbf{d}(t) = \mathbf{o}, \tag{38}$$

selecting, through (34), a position where the velocity is exactly $\mathbf{v}_o(t)$, parallel to $\mathbf{a}_o(t)$. The straight line passing through this position and parallel to $\mathbf{a}_o(t)$ is called *spin axis*.

3.1 Axial vector

Since $\mathbf{W}(t)$ transforms any vector $\mathbf{d}(t)$ into a vector orthogonal both to $\mathbf{d}(t)$ and $\mathbf{a}_o(t)$, there should be a vector $\boldsymbol{\omega}(t)$ such that

$$\mathbf{W}(t)\mathbf{d}(t) = \boldsymbol{\omega}(t) \times \mathbf{d}(t) \quad \forall \mathbf{d}(t) \in \mathcal{V}. \tag{39}$$

We can prove that it exists and is unique as follows. Since

$$\mathbf{W}(t)\boldsymbol{\omega}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{\omega}(t) = \mathbf{o}, \tag{40}$$

$\boldsymbol{\omega}(t)$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$. Hence it belongs to the same one-dimensional eigenspace as $\mathbf{a}_o(t)$. Setting for any given orthonormal basis

$$\boldsymbol{\omega}(t) = \omega_1(t)\mathbf{e}_1 + \omega_2(t)\mathbf{e}_2 + \omega_3(t)\mathbf{e}_3, \tag{41}$$

the components are obtained through (39)

$$\begin{aligned}
\mathbf{W}(t)\mathbf{e}_1 \cdot \mathbf{e}_2 &= \boldsymbol{\omega}(t) \times \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \times \mathbf{e}_2 \cdot \boldsymbol{\omega}(t) = \mathbf{e}_3 \cdot \boldsymbol{\omega}(t) \quad &\Rightarrow \quad \omega_3(t) = \mathbf{W}(t)\mathbf{e}_1 \cdot \mathbf{e}_2 \\
\mathbf{W}(t)\mathbf{e}_2 \cdot \mathbf{e}_3 &= \boldsymbol{\omega}(t) \times \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_2 \times \mathbf{e}_3 \cdot \boldsymbol{\omega}(t) = \mathbf{e}_1 \cdot \boldsymbol{\omega}(t) \quad &\Rightarrow \quad \omega_1(t) = \mathbf{W}(t)\mathbf{e}_2 \cdot \mathbf{e}_3 \\
\mathbf{W}(t)\mathbf{e}_3 \cdot \mathbf{e}_1 &= \boldsymbol{\omega}(t) \times \mathbf{e}_3 \cdot \mathbf{e}_1 = \mathbf{e}_3 \times \mathbf{e}_1 \cdot \boldsymbol{\omega}(t) = \mathbf{e}_2 \cdot \boldsymbol{\omega}(t) \quad &\Rightarrow \quad \omega_2(t) = \mathbf{W}(t)\mathbf{e}_3 \cdot \mathbf{e}_1
\end{aligned} \tag{42}$$

The vector $\boldsymbol{\omega}(t)$ is called *axial vector* of $\mathbf{W}(t)$. Relation (39) defines, through (42), an isomorphism between the space of skew symmetric tensors and the three-dimensional vector space $\mathcal{V}$.

3.2 Spin center

In a two-dimensional vector space, for a given rigid velocity field at time t, we call *spin center* the position of a point C such that $\dot{\mathsf{p}}_\mathsf{C}(t) = 0$. Since

$$\dot{\mathsf{p}}_\mathsf{A}(t) - \dot{\mathsf{p}}_\mathsf{C}(t) = \mathbf{W}(t)(\mathsf{p}_\mathsf{A}(t) - \mathsf{p}_\mathsf{C}(t)), \tag{43}$$

if $\mathsf{p}_\mathsf{C}(t)$ is the spin center then $\dot{\mathsf{p}}_\mathsf{A}(t)$ equals the velocity difference and hence it is orthogonal to the line joining the placements $\mathsf{p}_\mathsf{A}(t)$ and $\mathsf{p}_\mathsf{C}(t)$. That is why the spin center can be obtained by the intersection of lines drawn from any two positions and orthogonal to the corresponding velocities.

4 Rigid motions in coordinate form

Let us consider a two-dimensional Euclidean space and a Cartesian coordinate system defined by an origin O and an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. In a rigid motion the rotation $\mathbf{R}(t)$ at any time t can be described by

$$\mathbf{R}(t)\mathbf{e}_1 = \cos\theta(t)\mathbf{e}_1 + \sin\theta(t)\mathbf{e}_2, \tag{44}$$

$$\mathbf{R}(t)\mathbf{e}_2 = -\sin\theta(t)\mathbf{e}_1 + \cos\theta(t)\mathbf{e}_2. \tag{45}$$

Then the relation (10) can be transformed into the following one in terms of coordinates

$$\begin{pmatrix} x_{1\mathsf{B}}(t) \\ x_{2\mathsf{B}}(t) \end{pmatrix} = \begin{pmatrix} x_{1\mathsf{A}}(t) \\ x_{2\mathsf{A}}(t) \end{pmatrix} + \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \bar{x}_{1\mathsf{B}} - \bar{x}_{1\mathsf{A}} \\ \bar{x}_{2\mathsf{B}} - \bar{x}_{2\mathsf{A}} \end{pmatrix}. \tag{46}$$

Differentiating with respect to t we get

$$\begin{pmatrix} \dot{x}_{1B}(t) \\ \dot{x}_{2B}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_{1A}(t) \\ \dot{x}_{2A}(t) \end{pmatrix} + \dot{\theta}(t) \begin{pmatrix} -\sin\theta(t) & -\cos\theta(t) \\ \cos\theta(t) & -\sin\theta(t) \end{pmatrix} \begin{pmatrix} \bar{x}_{1B} - \bar{x}_{1A} \\ \bar{x}_{2B} - \bar{x}_{2A} \end{pmatrix}. \tag{47}$$

Then we can replace, from (46), the expression

$$\begin{pmatrix} \bar{x}_{1B} - \bar{x}_{1A} \\ \bar{x}_{2B} - \bar{x}_{2A} \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} x_{1B}(t) - x_{1A}(t) \\ x_{2B}(t) - x_{2A}(t) \end{pmatrix} \tag{48}$$

into (47), thus obtaining

$$\begin{pmatrix} \dot{x}_{1B}(t) \\ \dot{x}_{2B}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_{1A}(t) \\ \dot{x}_{2A}(t) \end{pmatrix} + \begin{pmatrix} 0 & -\dot{\theta}(t) \\ \dot{\theta}(t) & 0 \end{pmatrix} \begin{pmatrix} x_{1B}(t) - x_{1A}(t) \\ x_{2B}(t) - x_{2A}(t) \end{pmatrix}. \tag{49}$$

This is the expression relating the components of the velocities in (17). Hence the matrix in (49) is the matrix of $\mathbf{W}(t)$.

In general, in a three-dimensional Euclidean space endowed with a Cartesian coordinate system whose orthonormal vector basis is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the relation (17) can be transformed into the following one in terms of coordinates

$$\begin{pmatrix} \dot{x}_{1B}(t) \\ \dot{x}_{2B}(t) \\ \dot{x}_{3B}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_{1A}(t) \\ \dot{x}_{2A}(t) \\ \dot{x}_{3A}(t) \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} \begin{pmatrix} x_{1B}(t) - x_{1A}(t) \\ x_{2B}(t) - x_{2A}(t) \\ x_{3B}(t) - x_{3A}(t) \end{pmatrix}. \tag{50}$$

The matrix of $\mathbf{W}$ is skew symmetric because

$$\mathbf{W}^{\mathsf{T}} = -\mathbf{W} \quad \Rightarrow \quad \mathbf{W}\mathbf{e}_i \cdot \mathbf{e}_j = -\mathbf{e}_i \cdot \mathbf{W}\mathbf{e}_j. \tag{51}$$

5 Affine motion

A motion is said to be *affine* if the one-parameter family of deformations (4) is such that for any two body points A and B and for any t

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_B, t) = \boldsymbol{\phi}(\bar{\mathsf{p}}_A, t) + \mathbf{F}(t)(\bar{\mathsf{p}}_B - \bar{\mathsf{p}}_A), \tag{52}$$

where $\mathbf{F}(t)$ is a tensor such that $\det \mathbf{F}(t) > 0$. Hence an *affine motion* is defined by the motion of any body point, say A, and by the values of the *deformation gradient* $\mathbf{F}(t)$.

The velocity field in an affine motion is given by

$$\dot{\boldsymbol{\phi}}(\bar{\mathsf{p}}_B, t) = \dot{\boldsymbol{\phi}}(\bar{\mathsf{p}}_A, t) + \dot{\mathbf{F}}(t)(\bar{\mathsf{p}}_B - \bar{\mathsf{p}}_A). \tag{53}$$

Replacing (52) with

$$\mathsf{p}_B(t) = \mathsf{p}_A(t) + \mathbf{F}(t)(\bar{\mathsf{p}}_B - \bar{\mathsf{p}}_A), \tag{54}$$

we get

$$\dot{\mathsf{p}}_B(t) = \dot{\mathsf{p}}_A(t) + \dot{\mathbf{F}}(t)(\mathsf{p}_B(t_0) - \mathsf{p}_A(t_0)). \tag{55}$$

Since from (54)

$$\mathsf{p}_B(t_0) - \mathsf{p}_A(t_0) = \mathbf{F}(t)^{-1}(\mathsf{p}_B(t) - \mathsf{p}_A(t)), \tag{56}$$

then

$$\dot{\mathsf{p}}_{\mathsf{B}}(t) = \dot{\mathsf{p}}_{\mathsf{A}}(t) + \dot{\mathbf{F}}(t)\mathbf{F}(t)^{-1}(\mathsf{p}_{\mathsf{B}}(t) - \mathsf{p}_{\mathsf{A}}(t)). \tag{57}$$

Setting

$$\mathbf{L}(t) := \dot{\mathbf{F}}(t)\mathbf{F}(t)^{-1}, \tag{58}$$

the expression (57) can be written

$$\dot{\mathsf{p}}_{\mathsf{B}}(t) = \dot{\mathsf{p}}_{\mathsf{A}}(t) + \mathbf{L}(t)(\mathsf{p}_{\mathsf{B}}(t) - \mathsf{p}_{\mathsf{A}}(t)). \tag{59}$$

If we consider a deformation transforming positions at time t into positions at time $t+\tau$

$$\boldsymbol{\phi}_t(\mathsf{p}_{\mathsf{B}}(t),\tau) = \boldsymbol{\phi}_t(\mathsf{p}_{\mathsf{A}}(t),\tau) + \mathbf{F}_t(\tau)(\mathsf{p}_{\mathsf{B}}(t) - \mathsf{p}_{\mathsf{A}}(t)), \tag{60}$$

we get the following expression for the velocities at time t

$$\dot{\boldsymbol{\phi}}_t(\mathsf{p}_{\mathsf{B}}(t),0) = \dot{\boldsymbol{\phi}}_t(\mathsf{p}_{\mathsf{A}}(t),0) + \dot{\mathbf{F}}_t(0)(\mathsf{p}_{\mathsf{B}}(t) - \mathsf{p}_{\mathsf{A}}(t)). \tag{61}$$

By comparing this expression with (59) we obtain

$$\mathbf{L}(t) = \dot{\mathbf{F}}(t)\mathbf{F}(t)^{-1} = \dot{\mathbf{F}}_t(0). \tag{62}$$

This expression allows us to give a useful characterization of $\mathbf{L}$. From the polar decomposition of the deformation gradient

$$\mathbf{F}_t(\tau) = \mathbf{R}_t(\tau)\mathbf{U}_t(\tau) \tag{63}$$

we get, differentiating with respect to time at $\tau = 0$,

$$\mathbf{L}(t) = \dot{\mathbf{F}}_t(0) = \dot{\mathbf{R}}_t(0)\mathbf{U}_t(0) + \mathbf{R}_t(0)\dot{\mathbf{U}}_t(0) = \dot{\mathbf{R}}_t(0) + \dot{\mathbf{U}}_t(0), \tag{64}$$

since $\mathbf{F}_t(0) = \mathbf{I}$ implies $\mathbf{R}_t(0) = \mathbf{I}$ and $\mathbf{U}_t(0) = \mathbf{I}$. Note that $\dot{\mathbf{R}}_t(0)$ is skew symmetric while $\dot{\mathbf{U}}_t(0)$ is symmetric, since

$$\begin{aligned} \mathbf{R}_t(\tau)^{\mathsf{T}}\mathbf{R}_t(\tau) = \mathbf{I} \quad &\Rightarrow \quad \dot{\mathbf{R}}_t(\tau)^{\mathsf{T}}\mathbf{R}_t(\tau) + \mathbf{R}_t(\tau)^{\mathsf{T}}\dot{\mathbf{R}}_t(\tau) = \mathbf{O} \\ &\Rightarrow \quad \dot{\mathbf{R}}_t(0)^{\mathsf{T}} + \dot{\mathbf{R}}_t(0) = \mathbf{O}, && (65) \\ \dot{\mathbf{U}}_t(\tau)^{\mathsf{T}} = \dot{\mathbf{U}}_t(\tau) \quad &\Rightarrow \quad \dot{\mathbf{U}}_t(0)^{\mathsf{T}} = \dot{\mathbf{U}}_t(0). && (66) \end{aligned}$$

If we consider the decomposition of $\mathbf{L}(t)$

$$\mathbf{L}(t) = \mathbf{D}(t) + \mathbf{W}(t), \tag{67}$$

with

$$\mathbf{D}(t) := \frac{1}{2}(\mathbf{L}(t) + \mathbf{L}(t)^{\mathsf{T}}), \quad \mathbf{W}(t) := \frac{1}{2}(\mathbf{L}(t) - \mathbf{L}(t)^{\mathsf{T}}) \tag{68}$$

it turns out that

$$\mathbf{D}(t) = \dot{\mathbf{U}}_t(0), \quad \mathbf{W}(t) = \dot{\mathbf{R}}_t(0). \tag{69}$$

This is the reason why $\mathbf{D}(t)$ and $\mathbf{W}(t)$ are called *stretching* and *spin* respectively.

An *affine velocity field* is a velocity field whose velocities are given by (59), where $\mathbf{L}(t)$ is a tensor.

6 Velocity gradient

A generic motion at any time t can be described by a deformation

$$\boldsymbol{\phi}(\cdot, t) : \bar{\mathcal{R}} \mapsto \mathcal{E}. \tag{70}$$

Let us consider a straight line

$$\bar{c}(h) = \bar{\mathsf{p}}_{\mathsf{A}} + h\,\bar{\mathbf{d}} \tag{71}$$

on the shape $\bar{\mathcal{R}}$ and for each time t the curve $\boldsymbol{c}(\cdot, t)$ on $\mathcal{R}$ such that

$$\boldsymbol{c}(h, t) = \boldsymbol{\phi}(\bar{c}(h), t). \tag{72}$$

The tangent vector at $\boldsymbol{c}(0, t)$ is defined as the limit

$$\mathbf{d}(t) := \lim_{h\to 0} \frac{1}{h}\big(\boldsymbol{c}(h, t) - \boldsymbol{c}(0, t)\big). \tag{73}$$

At each time t we can define, for each body point A, the gradient of the vector field $\boldsymbol{\phi}(\cdot, t)$ as the tensor such that

$$\mathbf{F}(\bar{\mathsf{p}}_{\mathsf{A}}, t) : \bar{\mathbf{d}} \mapsto \mathbf{d}(t), \tag{74}$$

transforming vectors which are tangent to curves passing through $\bar{\mathsf{p}}_{\mathsf{A}}$ into vectors which are tangent to the corresponding curves through $\mathsf{p}_{\mathsf{A}}(t) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}, t)$.

At time t the velocity of body point is given by

$$\mathbf{v}_t : \mathsf{p}_{\mathsf{A}}(t) \mapsto \dot{\mathsf{p}}_{\mathsf{A}}(t). \tag{75}$$

whose domain is the shape of the body at time t and which is called *spatial velocity field*. The gradient of this vector field [see APPENDIX 2] is the tensor $\nabla\mathbf{v}_t$ such that

$$\nabla\mathbf{v}_t\mathbf{d}(t) = \lim_{h\to 0} \frac{1}{h}\big(\mathbf{v}_t(\boldsymbol{c}(h, t)) - \mathbf{v}_t(\boldsymbol{c}(0, t))\big) = \lim_{h\to 0} \frac{1}{h}\big(\dot{\boldsymbol{c}}(h, t) - \dot{\boldsymbol{c}}(0, t)\big). \tag{76}$$

Since from (73)

$$\dot{\mathbf{d}}(t) = \lim_{h\to 0} \frac{1}{h}\big(\dot{\boldsymbol{c}}(h, t) - \dot{\boldsymbol{c}}(0, t)\big), \tag{77}$$

we get from (76)

$$\nabla\mathbf{v}_t\mathbf{d}(t) = \dot{\mathbf{d}}(t). \tag{78}$$

From (74) we get also

$$\dot{\mathbf{d}}(t) = \lim_{\tau\to 0} \frac{1}{\tau}\big(\mathbf{d}(t+\tau) - \mathbf{d}(t)\big) = \dot{\mathbf{F}}(\bar{\mathsf{p}}_{\mathsf{A}}, t)\bar{\mathbf{d}} = \dot{\mathbf{F}}(\bar{\mathsf{p}}_{\mathsf{A}}, t)\mathbf{F}(\bar{\mathsf{p}}_{\mathsf{A}}, t)^{-1}\mathbf{d}(t). \tag{79}$$

Replacing this expression into (78) it turns out, dropping function arguments out,

$$\nabla\mathbf{v}_t = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{80}$$

If we consider the deformation from the shape at a fixed time t to any time $t+\tau$

$$\boldsymbol{\phi}_t(\cdot, \tau) : \mathsf{p}_{\mathsf{A}}(t) \mapsto \mathsf{p}_{\mathsf{A}}(t+\tau), \tag{81}$$

we can define, for each body point A, the gradient of $\boldsymbol{\phi}_t(\cdot, \tau)$, as the tensor

$$\mathbf{F}_t(\mathsf{p}_{\mathsf{A}}(t), \tau) : \mathbf{d}(t) \mapsto \mathbf{d}(t+\tau), \tag{82}$$

transforming vectors which are tangent to curves through $\mathsf{p}_A(t)$ into vectors tangent to the corresponding curves through $\mathsf{p}_A(t+\tau)$. Instead of (79) we get

$$\dot{\mathbf{d}}(t) = \lim_{\tau\to 0} \frac{1}{\tau}\big(\mathbf{d}(t+\tau) - \mathbf{d}(t)\big) = \dot{\mathbf{F}}_t(\mathsf{p}_A(t), t)\mathbf{d}(t) \tag{83}$$

and finally

$$\nabla \mathbf{v}_t = \dot{\mathbf{F}}_t. \tag{84}$$

7 Affine velocity fields

The meaning of the velocity gradient can be illustrated in the following way. In a two-dimensional space let us consider a body in the shape of a square at a time t. Let us consider also an orthonormal basis whose vectors are parallel to the sides of the square and denote the matrix of the velocity gradient by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{85}$$

The velocity field is described by (59). The shape the body takes in a sufficiently short time interval τ can be described by the expression

$$\begin{aligned} \mathsf{p}_B(t+\tau) &= \mathsf{p}_B(t) + \dot{\mathsf{p}}_B(t)\tau + \mathbf{o}(\tau) \\ &= \mathsf{p}_B(t) + \big(\dot{\mathsf{p}}_O(t) + \mathbf{L}(t)(\mathsf{p}_B(t) - \mathsf{p}_O(t))\big)\tau + \mathbf{o}(\tau) \end{aligned} \tag{86}$$

If we assume that the center is at rest ($\dot{\mathsf{p}}_O(t) = 0$) we get

$$\mathsf{p}_B(t+\tau) = \mathsf{p}_B(t) + \mathbf{L}(t)(\mathsf{p}_B(t) - \mathsf{p}_O(t))\tau + \mathbf{o}(\tau). \tag{87}$$

Figures 1, 2, 3, show the shapes the body takes according to the values of $\mathbf{L}(t)$ given by the matrices in the tables below arranged in the same order as the shapes

Fig. 1	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Fig. 2	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$
	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Fig. 3	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$
	$\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

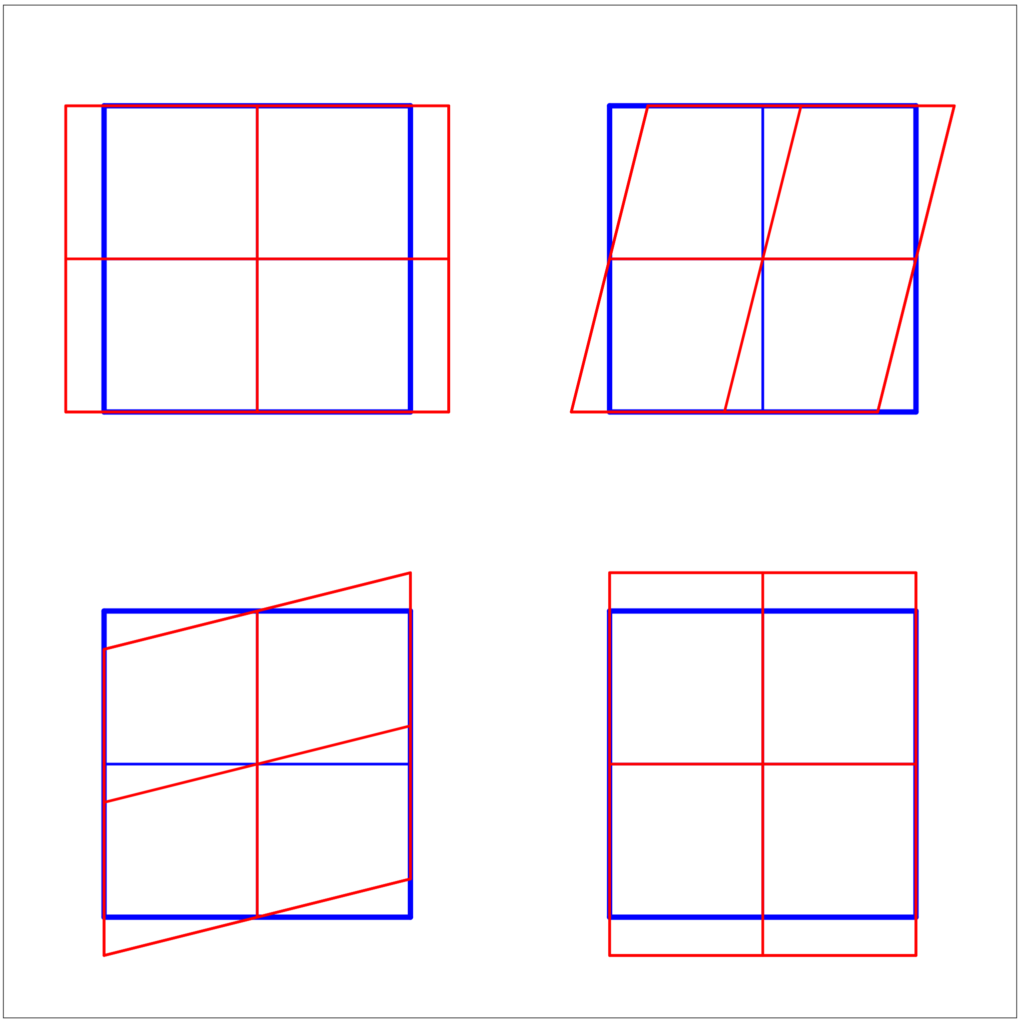

Figure 1: Illustration of the velocity gradient

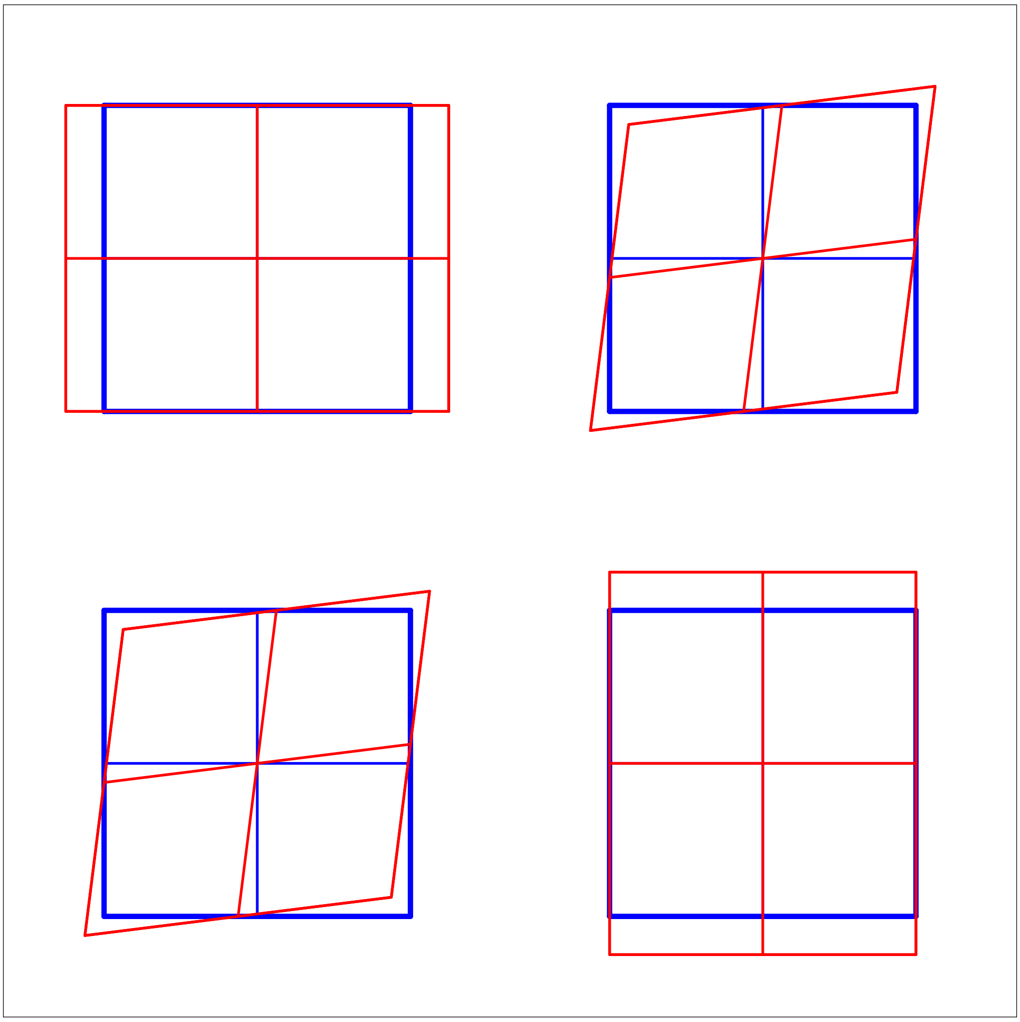

Figure 2: Illustration of the velocity gradient (symmetric part)

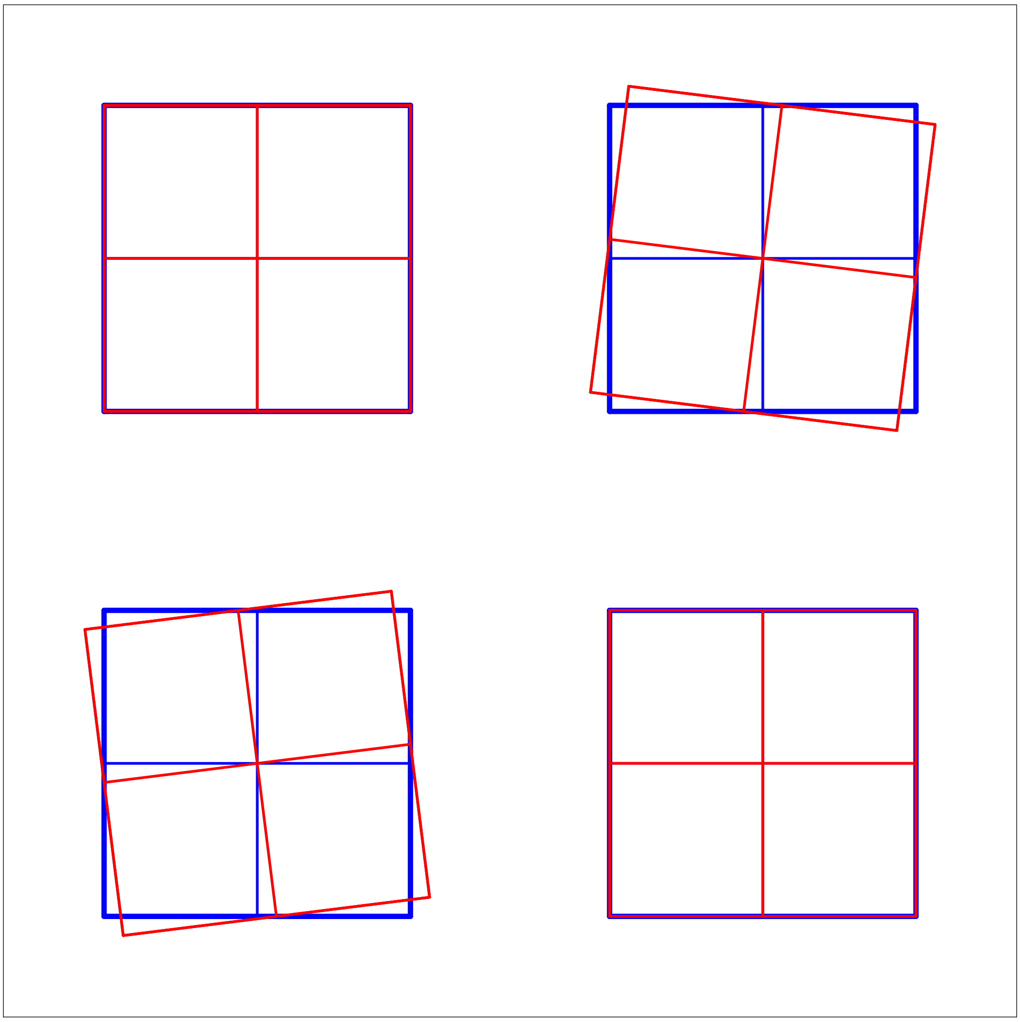

Figure 3: Illustration of the velocity gradient (skew symmetric part)