Power, Forces and Moments

Mais il y a aussi, au moins depuis d'Alambert, une deuxième voie possible, celle des puissances (ou travaux) virtuelles. Contrairement à ce que l'on croit parfois, cette deuxième manière est tout aussi naturelle que la première et elle ne fait che traduire une expérience physique très commune. Si on veut savoir si une valise è lourde, on essaie de la soulever un peu; pour apprècier la tension d'une courroie de transmission, on l'écarte quelque peu de sa position stable; et c'est en essayant de pousser une voiture que l'on se rendra compte des frottements tant externes qu'internes s'opposant au mouvement.

[...] L'idée essentielle de cette deuxième voie est celle de "dualité". Aussi cette voie est-elle non seulement très proche de l'expérience la plus commune comme nous l'avons déjà noté, mais aussi trés souple; selon que l'on choisira un espace vectoriel plus ou moins "vaste", on aura une description des efforts plus ou moins fine.

[Germain, P., La méthode des puissances virtuelles en mécanique des milieux continus. Première partie: Théorie du second gradient, *Journal de Mécanique*, 12 (1973), pp. 235–274.]

Contents

1 Power and forces

Let us consider a body $\mathcal{B} := \{\mathsf{A}, \mathsf{B}\}$, made up of two points and a placement

$$\mathsf{p} : \mathcal{B} \to \mathcal{E}. \tag{1}$$

The simplest way of describing the *mechanical interaction* of the body and its environment is to give a linear function $\mathcal{W}^{(ext)}$, called *power*, which for any placement p transforms any velocity field into a scalar.

The choice of the *test velocity space* plays a fundamental role in the modeling. If we denote the test velocities of body points p_A and p_B by

$$\mathbf{v}_\mathsf{A}, \mathbf{v}_\mathsf{B} \tag{2}$$

the power can be uniquely given the form

$$\mathcal{W}^{(ext)}(\mathbf{v}_\mathsf{A}, \mathbf{v}_\mathsf{B}) = \mathbf{f}_\mathsf{A} \cdot \mathbf{v}_\mathsf{A} + \mathbf{f}_\mathsf{B} \cdot \mathbf{v}_\mathsf{B}. \tag{3}$$

Vectors $\mathbf{f}_\mathsf{A}, \mathbf{f}_\mathsf{B}$ are called the *forces* applied respectively to A and B.

The most interesting case is when $\mathcal{B}$ and p are such that the set $\operatorname{im} \mathsf{p}$ (the *shape* of the body) is a subset $\mathcal{R} \subset \mathcal{E}$ which is the closure of an open set. A test velocity field is a function

$$\mathbf{v} : \mathsf{p}_\mathsf{A} \mapsto \mathbf{v}_\mathsf{A}, \tag{4}$$

whose domain is $\mathcal{R}$. The *exterior power* can be given in general the following representation

$$\mathcal{W}^{(ext)}(\mathbf{v}) = \int_\mathcal{R} \mathbf{b} \cdot \mathbf{v}\, dV + \int_{\partial\mathcal{R}} \mathbf{t} \cdot \mathbf{v}\, dA. \tag{5}$$

The vector fields $\mathbf{b}$ and $\mathbf{t}$, respectively on $\mathcal{R}$ and $\partial\mathcal{R}$, are called *bulk force distribution* and *surface force distributions (or contact forces)*.

2 Power for a rigid test velocity field

2.1 Forces and moments

In a rigid test velocity field the velocities are such that

$$\begin{aligned} \mathbf{v}_\mathsf{A} &= \mathbf{v}_\mathsf{O} + \mathbf{W}(\mathsf{p}_\mathsf{A} - \mathsf{p}_\mathsf{O}), \\ \mathbf{v}_\mathsf{B} &= \mathbf{v}_\mathsf{O} + \mathbf{W}(\mathsf{p}_\mathsf{B} - \mathsf{p}_\mathsf{O}). \end{aligned} \tag{6}$$

Hence the exterior power (3) becomes

$$\begin{aligned} \mathcal{W}^{(ext)}(\mathbf{v}_\mathsf{A}, \mathbf{v}_\mathsf{B}) &= \mathbf{f}_\mathsf{A} \cdot \mathbf{v}_\mathsf{A} + \mathbf{f}_\mathsf{B} \cdot \mathbf{v}_\mathsf{B} \\ &= (\mathbf{f}_\mathsf{A} + \mathbf{f}_\mathsf{B}) \cdot \mathbf{v}_\mathsf{O} + \mathbf{f}_\mathsf{A} \cdot \mathbf{W}(\mathsf{p}_\mathsf{A} - \mathsf{p}_\mathsf{O}) + \mathbf{f}_\mathsf{B} \cdot \mathbf{W}(\mathsf{p}_\mathsf{B} - \mathsf{p}_\mathsf{O}) \\ &= (\mathbf{f}_\mathsf{A} + \mathbf{f}_\mathsf{B}) \cdot \mathbf{v}_\mathsf{O} + (\mathsf{p}_\mathsf{A} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f}_\mathsf{A} \cdot \mathbf{W} + (\mathsf{p}_\mathsf{B} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f}_\mathsf{B} \cdot \mathbf{W} \\ &= (\mathbf{f}_\mathsf{A} + \mathbf{f}_\mathsf{B}) \cdot \mathbf{v}_\mathsf{O} + \big((\mathsf{p}_\mathsf{A} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f}_\mathsf{A} + (\mathsf{p}_\mathsf{B} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f}_\mathsf{B}\big) \cdot \mathbf{W}. \end{aligned} \tag{7}$$

by using the axial vector $\boldsymbol{\omega}$ of the spin $\mathbf{W}$, expressions (6) turn into

$$\begin{aligned} \mathbf{v}_\mathsf{A} &= \mathbf{v}_\mathsf{O} + \boldsymbol{\omega} \times (\mathsf{p}_\mathsf{A} - \mathsf{p}_\mathsf{O}), \\ \mathbf{v}_\mathsf{B} &= \mathbf{v}_\mathsf{O} + \boldsymbol{\omega} \times (\mathsf{p}_\mathsf{B} - \mathsf{p}_\mathsf{O}), \end{aligned} \tag{8}$$

which give the following expression for the power

$$\mathcal{W}^{(ext)}(\mathbf{v}_\mathsf{A},\mathbf{v}_\mathsf{B}) = (\mathbf{f}_\mathsf{A}+\mathbf{f}_\mathsf{B})\cdot\mathbf{v}_\mathsf{O} + \big((\mathsf{p}_\mathsf{A}-\mathsf{p}_\mathsf{O})\times\mathbf{f}_\mathsf{A} + (\mathsf{p}_\mathsf{B}-\mathsf{p}_\mathsf{O})\times\mathbf{f}_\mathsf{B}\big)\cdot\boldsymbol{\omega}. \tag{9}$$

The coefficient of $\mathbf{v}_\mathsf{O}$

$$\mathbf{f} := \mathbf{f}_\mathsf{A}+\mathbf{f}_\mathsf{B} \tag{10}$$

is called *total force*, the coefficient of $\boldsymbol{\omega}$

$$\mathbf{m}_{\mathsf{p}_\mathsf{O}} := (\mathsf{p}_\mathsf{A}-\mathsf{p}_\mathsf{O})\times\mathbf{f}_\mathsf{A} + (\mathsf{p}_\mathsf{B}-\mathsf{p}_\mathsf{O})\times\mathbf{f}_\mathsf{B} \tag{11}$$

is called *total moment vector with respect to* p_O. The coefficient of $\mathbf{W}$

$$\mathbf{M}_{\mathsf{p}_\mathsf{O}} := (\mathsf{p}_\mathsf{A}-\mathsf{p}_\mathsf{O})\otimes\mathbf{f}_\mathsf{A} + (\mathsf{p}_\mathsf{B}-\mathsf{p}_\mathsf{O})\otimes\mathbf{f}_\mathsf{B} \tag{12}$$

is called *total moment tensor with respect to* p_O.

Note that (7), because of the skew symmetry of $\mathbf{W}$, can be written also

$$\mathcal{W}^{(ext)}(\mathbf{v}_\mathsf{A},\mathbf{v}_\mathsf{B}) = (\mathbf{f}_\mathsf{A}+\mathbf{f}_\mathsf{B})\cdot\mathbf{v}_\mathsf{O} + \operatorname{skw}\big((\mathsf{p}_\mathsf{A}-\mathsf{p}_\mathsf{O})\otimes\mathbf{f}_\mathsf{A} + (\mathsf{p}_\mathsf{B}-\mathsf{p}_\mathsf{O})\otimes\mathbf{f}_\mathsf{B}\big)\cdot\mathbf{W}. \tag{13}$$

For a body $\mathcal{B}$ in the shape $\mathcal{R}$, any rigid test velocity field can be described by

$$\mathbf{v}(\boldsymbol{x}) = \mathbf{v}_\mathsf{O} + \mathbf{W}(\boldsymbol{x}-\mathsf{p}_\mathsf{O}) \quad \forall\boldsymbol{x}\in\mathcal{R}. \tag{14}$$

The exterior power (5) becomes

$$\begin{aligned}\mathcal{W}^{(ext)}(\mathbf{v}) &= \int_\mathcal{R}\mathbf{b}\cdot\mathbf{v}\,dV + s\int_{\partial\mathcal{R}}\mathbf{t}\cdot\mathbf{v}\,dA\\ &= \Big(\int_\mathcal{R}\mathbf{b}\,dV + \int_{\partial\mathcal{R}}\mathbf{t}\,dA\Big)\cdot\mathbf{v}_\mathsf{O} + \Big(\int_\mathcal{R}(\boldsymbol{x}-\mathsf{p}_\mathsf{O})\otimes\mathbf{b}\,dV + \int_{\partial\mathcal{R}}(\boldsymbol{x}-\mathsf{p}_\mathsf{O})\otimes\mathbf{t}\,dA\Big)\cdot\mathbf{W}. \end{aligned}\tag{15}$$

In such a case the total force and the total moment tensor are

$$\mathbf{f} := \int_\mathcal{R}\mathbf{b}\,dV + \int_{\partial\mathcal{R}}\mathbf{t}\,dA, \tag{16}$$

$$\mathbf{M}_{\mathsf{p}_\mathsf{O}} := \int_\mathcal{R}(\boldsymbol{x}-\mathsf{p}_\mathsf{O})\otimes\mathbf{b}\,dV + \int_{\partial\mathcal{R}}(\boldsymbol{x}-\mathsf{p}_\mathsf{O})\otimes\mathbf{t}\,dA, \tag{17}$$

while (15) reads

$$\mathcal{W}^{(ext)}(\mathbf{v}) = \mathbf{f}\cdot\mathbf{v}_\mathsf{O} + \mathbf{M}_{\mathsf{p}_\mathsf{O}}\cdot\mathbf{W}. \tag{18}$$

As an alternative, by using the axial vector of $\mathbf{W}$ we can obtain the total moment tensor

$$\mathbf{m}_{\mathsf{p}_\mathsf{O}} := \int_\mathcal{R}(\boldsymbol{x}-\mathsf{p}_\mathsf{O})\times\mathbf{b}\,dV + \int_{\partial\mathcal{R}}(\boldsymbol{x}-\mathsf{p}_\mathsf{O})\times\mathbf{t}\,dA, \tag{19}$$

while (15) reads

$$\mathcal{W}^{(ext)}(\mathbf{v}) = \mathbf{f}\cdot\mathbf{v}_\mathsf{O} + \mathbf{m}_{\mathsf{p}_\mathsf{O}}\cdot\boldsymbol{\omega}. \tag{20}$$

2.2 Moment vector and skew symmetric part of the moment tensor

Note that the *moment vector* is the axial vector of $2\,\text{skw}\,\mathbf{M}$. By comparing (18) and (20) we get

$$\mathbf{M}\cdot\mathbf{W} = \mathbf{m}\cdot\boldsymbol{\omega} \tag{21}$$

where $\boldsymbol{\omega}$ is the axialvector of $\mathbf{W}$. Since

$$\begin{aligned}\mathbf{M}\cdot\mathbf{W} &= \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ &= (M_{32}-M_{23})\omega_1 + (M_{13}-M_{31})\omega_2 + (M_{21}-M_{12})\omega_3, \end{aligned} \tag{22}$$

$$\begin{aligned}\text{skw}\,\mathbf{M}\cdot\mathbf{W} &= \frac{1}{2}\begin{pmatrix} 0 & M_{12}-M_{21} & M_{13}-M_{31} \\ M_{21}-M_{12} & 0 & M_{23}-M_{32} \\ M_{31}-M_{13} & M_{32}-M_{23} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ &= (M_{32}-M_{23})\omega_1 + (M_{13}-M_{31})\omega_2 + (M_{21}-M_{12})\omega_3, \end{aligned} \tag{23}$$

$$\begin{aligned}\mathbf{m}\cdot\boldsymbol{\omega} &= (m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3)\cdot(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3) \\ &= m_1\omega_1 + m_2\omega_2 + m_3\omega_3, \end{aligned} \tag{24}$$

the components of $\mathbf{m}$ turns out to be such that

$$\begin{aligned} m_1 &= M_{32} - M_{23}, \\ m_2 &= M_{13} - M_{31}, \\ m_3 &= M_{21} - M_{12}. \end{aligned} \tag{25}$$

Hence they are equal respectively to terms $(3,2)$, $(1,3)$ and $(2,1)$ of the matrix of $\mathbf{M}-\mathbf{M}^{\mathsf{T}} = 2\,\text{skw}\,\mathbf{M}$.

3 Power for an affine test velocity field

3.1 Forces and moments

An affine test velocity field is such that

$$\begin{aligned} \mathbf{v}_A &= \mathbf{v}_O + \mathbf{L}(p_A - p_O), \\ \mathbf{v}_B &= \mathbf{v}_O + \mathbf{L}(p_B - p_O). \end{aligned} \tag{26}$$

Hence the exterior power becomes

$$\begin{aligned} \mathcal{W}^{(ext)}(\mathbf{v}_A,\mathbf{v}_B) &= \mathbf{f}_A\cdot\mathbf{v}_A + \mathbf{f}_B\cdot\mathbf{v}_B \\ &= (\mathbf{f}_A+\mathbf{f}_B)\cdot\mathbf{v}_O + \big(\mathbf{f}_A\cdot\mathbf{L}(p_A-p_O) + \mathbf{f}_B\cdot\mathbf{L}(p_B-p_O)\big) \\ &= (\mathbf{f}_A+\mathbf{f}_B)\cdot\mathbf{v}_O + \big((p_A-p_O)\otimes\mathbf{f}_A + (p_B-p_O)\otimes\mathbf{f}_B\big)\cdot\mathbf{L} \\ &= (\mathbf{f}_A+\mathbf{f}_B)\cdot\mathbf{v}_O + \text{skw}\,\big((p_A-p_O)\otimes\mathbf{f}_A + (p_B-p_O)\otimes\mathbf{f}_B\big)\cdot\mathbf{W} \\ &\qquad + \text{sym}\,\big((p_A-p_O)\otimes\mathbf{f}_A + (p_B-p_O)\otimes\mathbf{f}_B\big)\cdot\mathbf{D}. \end{aligned} \tag{27}$$

The coefficient of $\mathbf{L}$ is called the *total moment tensor*

$$\mathbf{M}_{\mathsf{p}_\mathsf{O}} := (\mathsf{p}_\mathsf{A} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f}_\mathsf{A} + (\mathsf{p}_\mathsf{B} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f}_\mathsf{B}. \tag{28}$$

The coefficient of $\mathbf{W}$, the skew symmetric part of $\mathbf{L}$, can be called the total *skew symmetric moment* while the coefficient of $\mathbf{D}$, the symmetric part of $\mathbf{L}$, can be called the total *symmetric moment*, with respect to p_O.

For a body $\mathcal{B}$ in the shape $\mathcal{R}$, an affine test velocity field can be described by the function

$$\mathbf{v}(x) = \mathbf{v}_\mathsf{O} + \mathbf{L}(x - \mathsf{p}_\mathsf{O}) \quad \forall x \in \mathcal{R}. \tag{29}$$

The exterior power (5) becomes

$$\begin{aligned} \mathcal{W}^{(ext)}(\mathbf{v}) &= \int_\mathcal{R} \mathbf{b} \cdot \mathbf{v}\, dV + \int_{\partial\mathcal{R}} \mathbf{t} \cdot \mathbf{v}\, dA \\ &= \Big(\int_\mathcal{R} \mathbf{b}\, dV + \int_{\partial\mathcal{R}} \mathbf{t}\, dA \Big) \cdot \mathbf{v}_\mathsf{O} + \Big(\int_\mathcal{R} (x - \mathsf{p}_\mathsf{O}) \otimes \mathbf{b}\, dV + \int_{\partial\mathcal{R}} (x - \mathsf{p}_\mathsf{O}) \otimes \mathbf{t}\, dA \Big) \cdot \mathbf{L}. \end{aligned} \tag{30}$$

The total force and the total moment tensor are

$$\mathbf{f} := \int_\mathcal{R} \mathbf{b}\, dV + \int_{\partial\mathcal{R}} \mathbf{t}\, dA, \tag{31}$$

$$\mathbf{M}_{\mathsf{p}_\mathsf{O}} := \int_\mathcal{R} (x - \mathsf{p}_\mathsf{O}) \otimes \mathbf{b}\, dV + \int_{\partial\mathcal{R}} (x - \mathsf{p}_\mathsf{O}) \otimes \mathbf{t}\, dA \tag{32}$$

while (30) reads

$$\mathcal{W}^{(ext)}(\mathbf{v}) = \mathbf{f} \cdot \mathbf{v}_\mathsf{O} + \mathbf{M}_{\mathsf{p}_\mathsf{O}} \cdot \mathbf{L}. \tag{33}$$

3.2 Moment with respect to a different position

Let us consider a different description of the test velocity field (29)

$$\mathbf{v}(x) = \mathbf{v}_\mathsf{C} + \mathbf{L}(x - \mathsf{p}_\mathsf{C}), \quad \forall x \in \mathcal{R}. \tag{34}$$

Since

$$\mathbf{v}_\mathsf{C} = \mathbf{v}_\mathsf{O} + \mathbf{L}(\mathsf{p}_\mathsf{C} - \mathsf{p}_\mathsf{O}) \tag{35}$$

we get

$$\mathbf{v}(x) = \mathbf{v}_\mathsf{O} + \mathbf{L}(\mathsf{p}_\mathsf{C} - \mathsf{p}_\mathsf{O}) + \mathbf{L}(x - \mathsf{p}_\mathsf{C}) = \mathbf{v}_\mathsf{O} + \mathbf{L}(x - \mathsf{p}_\mathsf{O}). \tag{36}$$

Then, through (30), we arrive at the the following relation between the moment tensor with respect to p_C and the moment tensor with respect to p_O

$$\begin{aligned} &(\mathsf{p}_\mathsf{C} - \mathsf{p}_\mathsf{O}) \otimes \Big(\int_\mathcal{R} \mathbf{b}\, dV + \int_{\partial\mathcal{R}} \mathbf{t}\, dA \Big) + \Big(\int_\mathcal{R} (x - \mathsf{p}_\mathsf{C}) \otimes \mathbf{b}\, dV + \int_{\partial\mathcal{R}} (x - \mathsf{p}_\mathsf{C}) \otimes \mathbf{t}\, dA \Big) \\ &= \Big(\int_\mathcal{R} (x - \mathsf{p}_\mathsf{O}) \otimes \mathbf{b}\, dV + \int_{\partial\mathcal{R}} (x - \mathsf{p}_\mathsf{O}) \otimes \mathbf{t}\, dA \Big). \end{aligned} \tag{37}$$

Denoting by $\mathbf{f}$ the total force, by $\mathbf{M}_{\mathsf{p}_\mathsf{O}}$ the total moment tensor with respect to p_O and by $\mathbf{M}_{\mathsf{p}_\mathsf{C}}$ the total moment tensor with respect to p_C, from the above expression we get

$$(\mathsf{p}_\mathsf{C} - \mathsf{p}_\mathsf{O}) \otimes \mathbf{f} + \mathbf{M}_{\mathsf{p}_\mathsf{C}} = \mathbf{M}_{\mathsf{p}_\mathsf{O}}. \tag{38}$$

Note that if the total force is zero then the moment tensor is independent of the choice of p_O.

4 Equipowerful force distributions

We call *equipowerful* those force distributions spending the same power on each test velocity field.

When considering rigid test velocity fields, equipowerful force distributions share the same total force and the same moment vector (or the skew symmetric moment tensor).

When considering affine test velocity fields, equipowerful force distributions share the same total force and the same total moment tensor (both symmetric and skew symmetric part).

It is worth noting that if two different force distributions have the same total force and the same total moment tensor with respect a position p_O, they still have the same moment, possibly different from the previous one, with respect to any other position by (38).

4.1 Forces applied at the face centres of a rectangular parallelepiped

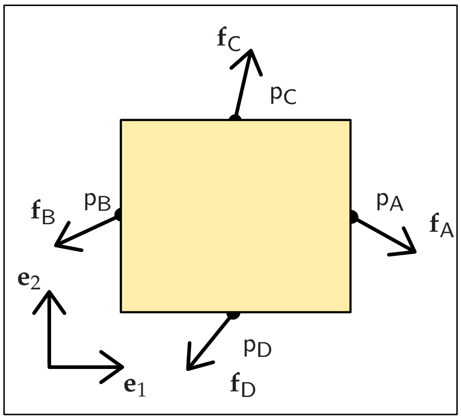

Figure 1: Forces applied at the face centres of a rectangular parallelepiped.

Let us consider a body in the shape of a rectangular parallelepiped, whose edge lengths are ℓ_1, ℓ_2, ℓ_3, with forces applied at the center of the faces orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_2$, like in fig. 1. The total force is

$$\mathbf{f} = \mathbf{f}_A + \mathbf{f}_B + \mathbf{f}_C + \mathbf{f}_D. \tag{39}$$

The total moment tensor, with respect to the parallelepiped center p_O, is

$$\begin{aligned} \mathbf{M}_{p_O} &= (p_A - p_O) \otimes \mathbf{f}_A + (p_B - p_O) \otimes \mathbf{f}_B + (p_C - p_O) \otimes \mathbf{f}_C + (p_D - p_O) \otimes \mathbf{f}_D \\ &= \ell_1 \mathbf{e}_1 \otimes \frac{1}{2}(\mathbf{f}_A - \mathbf{f}_B) + \ell_2 \mathbf{e}_2 \otimes \frac{1}{2}(\mathbf{f}_C - \mathbf{f}_D). \end{aligned} \tag{40}$$

4.2 Uniform force distribution on the faces of a rectangular parallelepiped

Let us consider the force distribution in fig. 2, which is uniform on each of the faces orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_2$. The barycenter $\boldsymbol{c}_{\mathcal{F}}$ of a face $\mathcal{F}$ is defined by the property

$$(\boldsymbol{c}_{\mathcal{F}} - p_O) A_{\mathcal{F}} = \int_{\mathcal{F}} (x - p_O)\, dA\,, \tag{41}$$

where $A_{\mathcal{F}}$ is the area of face $\mathcal{F}$ and p_O is an arbitrary position, even outside $\mathcal{F}$. Since the barycenters of rectangular faces are their centres, choosing p_O as the parallelpiped center, the moment

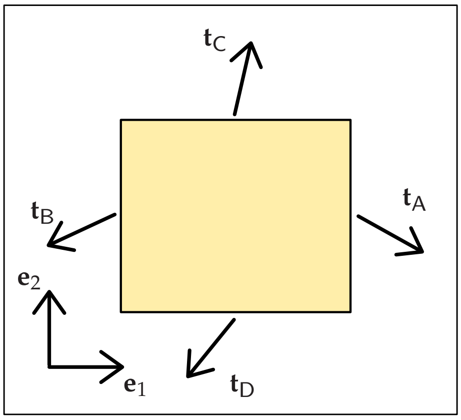

Figure 2: Uniform force distribution on the faces of a rectangular parallelepiped.

tensor of the force distribution on $\mathcal{F}_1$ and $\mathcal{F}_{-1}$ is

$$\begin{aligned}&\int_{\mathcal{F}_1}(x-\mathsf{p}_O)\otimes\mathbf{t}_A\,dA+\int_{\mathcal{F}_{-1}}(x-\mathsf{p}_O)\otimes\mathbf{t}_B\,dA\\&=A_{\mathcal{F}_1}\Big(\frac{\ell_1}{2}\mathbf{e}_1\Big)\otimes\mathbf{t}_A+A_{\mathcal{F}_{-1}}\Big(-\frac{\ell_1}{2}\mathbf{e}_1\Big)\otimes\mathbf{t}_B=\frac{\ell_1}{2}A_{\mathcal{F}_1}\,\mathbf{e}_1\otimes(\mathbf{t}_A-\mathbf{t}_B)\end{aligned}\tag{42}$$

while the moment tensor of the force distribution on $\mathcal{F}_2$ e $\mathcal{F}_{-2}$ is

$$\begin{aligned}&\int_{\mathcal{F}_2}(x-\mathsf{p}_O)\otimes\mathbf{t}_C\,dA+\int_{\mathcal{F}_{-2}}(x-\mathsf{p}_O)\otimes\mathbf{t}_D\,dA\\&=A_{\mathcal{F}_2}\Big(\frac{\ell_2}{2}\mathbf{e}_2\Big)\otimes\mathbf{t}_C+A_{\mathcal{F}_{-2}}\Big(-\frac{\ell_2}{2}\mathbf{e}_2\Big)\otimes\mathbf{t}_D=\frac{\ell_2}{2}A_{\mathcal{F}_2}\,\mathbf{e}_2\otimes(\mathbf{t}_C-\mathbf{t}_D)\,.\end{aligned}\tag{43}$$

By using the area function induced by the volume function (see APPENDIX 2), we get

$$\begin{aligned}&\int_{\mathcal{F}_1}dA=A_{\mathcal{F}_1}=\mathrm{vol}(\ell_2\,\mathbf{e}_2,\ell_3\,\mathbf{e}_3,\mathbf{e}_1)=\frac{1}{\ell_1}\,\mathrm{vol}(\ell_2\,\mathbf{e}_2,\ell_3\,\mathbf{e}_3,\ell_1\,\mathbf{e}_1)=\frac{1}{\ell_1}V_{\mathcal{R}}\,,\\&\int_{\mathcal{F}_2}dA=A_{\mathcal{F}_2}=\mathrm{vol}(\ell_3\,\mathbf{e}_3,\ell_1\,\mathbf{e}_1,\mathbf{e}_2)=\frac{1}{\ell_2}\,\mathrm{vol}(\ell_3\,\mathbf{e}_3,\ell_1\,\mathbf{e}_1,\ell_2\,\mathbf{e}_2)=\frac{1}{\ell_2}V_{\mathcal{R}}\,,\end{aligned}\tag{44}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are normal unit vectors to the faces $\mathcal{F}_1$ and $\mathcal{F}_2$. Hence the total moment tensor will be

$$\mathbf{M}_{\mathsf{p}_O}=V_{\mathcal{R}}\Big(\mathbf{e}_1\otimes\frac{1}{2}(\mathbf{t}_A-\mathbf{t}_B)+\mathbf{e}_2\otimes\frac{1}{2}(\mathbf{t}_C-\mathbf{t}_D)\Big)\tag{45}$$

The total force turns out to be

$$\mathbf{f}=A_{\mathcal{F}_1}(\mathbf{t}_A+\mathbf{t}_B)+A_{\mathcal{F}_2}(\mathbf{t}_C+\mathbf{t}_D).\tag{46}$$

Note that by choosing

$$\mathbf{t}_A=\frac{\mathbf{f}_A}{A_{\mathcal{F}_1}},\quad\mathbf{t}_B=\frac{\mathbf{f}_B}{A_{\mathcal{F}_1}},\quad\mathbf{t}_C=\frac{\mathbf{f}_C}{A_{\mathcal{F}_2}},\quad\mathbf{t}_D=\frac{\mathbf{f}_D}{A_{\mathcal{F}_2}},\tag{47}$$

the force distribution just considered turns out to belong to the same equipowerful class as the previous one.

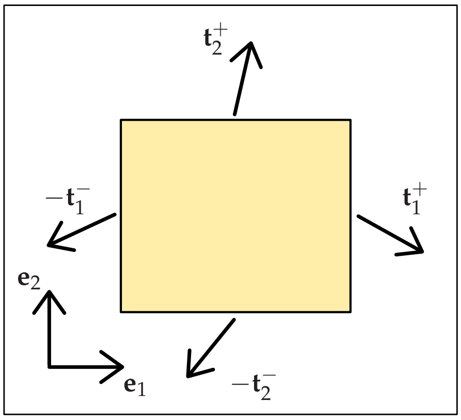

Figure 3: Uniform force distribution on the faces of a parallalepiped.

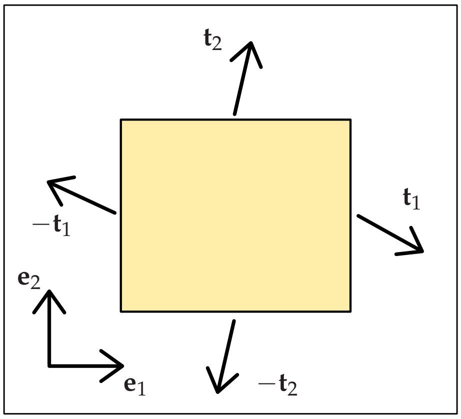

Figure 4: Uniform force distribution on the faces of a parallalepiped, with opposite values on opposite faces.

4.3 Decomposition of a force distribution on the faces of a rectangular parallelepiped

Let us consider once again the force distribution in fig. 2 while changing names according to fig. 3 and adding a force distribution on the faces orthogonal to $\mathbf{e}_3$ as well. We can define a different force distribution, as in fig. 4, made up of a uniform distribution on the boundary with opposite values on opposite faces, by

$$\mathbf{t}_1 := \frac{1}{2}(\mathbf{t}_1^+ + \mathbf{t}_1^-), \quad \mathbf{t}_2 := \frac{1}{2}(\mathbf{t}_2^+ + \mathbf{t}_2^-), \quad \mathbf{t}_3 := \frac{1}{2}(\mathbf{t}_3^+ + \mathbf{t}_3^-), \tag{48}$$

The expression for the moment tensor with respect to the center turns out to be, from that computed for the force distribution in fig. 2,

$$\begin{aligned} \mathbf{M}_{p_O} &= V_{\mathcal{R}}\Big(\mathbf{e}_1 \otimes \frac{1}{2}(\mathbf{t}_1^+ + \mathbf{t}_1^-) + \mathbf{e}_2 \otimes \frac{1}{2}(\mathbf{t}_2^+ + \mathbf{t}_2^-) + \mathbf{e}_3 \otimes \frac{1}{2}(\mathbf{t}_3^+ + \mathbf{t}_3^-)\Big) \\ &= V_{\mathcal{R}}\Big(\mathbf{e}_1 \otimes \mathbf{t}_1 + \mathbf{e}_2 \otimes \mathbf{t}_2 + \mathbf{e}_3 \otimes \mathbf{t}_3\Big), \end{aligned} \tag{49}$$

where p_O is again the center of the parallelepiped.. Note that (49) implies

$$\begin{aligned}\mathbf{M}_{\mathsf{p}_O}\mathbf{e}_1 &= V_{\mathcal{R}}\,\mathbf{t}_1,\\ \mathbf{M}_{\mathsf{p}_O}\mathbf{e}_2 &= V_{\mathcal{R}}\,\mathbf{t}_2,\\ \mathbf{M}_{\mathsf{p}_O}\mathbf{e}_3 &= V_{\mathcal{R}}\,\mathbf{t}_3.\end{aligned} \tag{50}$$

By setting

$$\mathbf{t}_1 = t_{11}\mathbf{e}_1 + t_{21}\mathbf{e}_2 + t_{31}\mathbf{e}_3, \quad \mathbf{t}_2 = t_{12}\mathbf{e}_1 + t_{22}\mathbf{e}_2 + t_{32}\mathbf{e}_3, \quad \mathbf{t}_2 = t_{13}\mathbf{e}_1 + t_{23}\mathbf{e}_2 + t_{33}\mathbf{e}_3, \tag{51}$$

we get the matrix of the moment tensor

$$[\mathbf{M}_{\mathsf{p}_O}] = V_{\mathcal{R}}\begin{pmatrix} t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{pmatrix}, \tag{52}$$

Further, by defining

$$\mathbf{t}'_1 := \frac{1}{\ell_1}(\mathbf{t}_1^+ - \mathbf{t}_1^-), \quad \mathbf{t}'_2 := \frac{1}{\ell_2}(\mathbf{t}_2^+ - \mathbf{t}_2^-), \quad \mathbf{t}'_3 := \frac{1}{\ell_3}(\mathbf{t}_3^+ - \mathbf{t}_3^-), \tag{53}$$

we can derive the following decomposition

$$\begin{aligned}\mathbf{t}_1^+ &= \mathbf{t}_1 + \frac{\ell_1}{2}\mathbf{t}'_1, \quad \mathbf{t}_1^- = \mathbf{t}_1 - \frac{\ell_1}{2}\mathbf{t}'_1,\\ \mathbf{t}_2^+ &= \mathbf{t}_2 + \frac{\ell_2}{2}\mathbf{t}'_2, \quad \mathbf{t}_2^- = \mathbf{t}_2 - \frac{\ell_2}{2}\mathbf{t}'_2,\\ \mathbf{t}_3^+ &= \mathbf{t}_3 + \frac{\ell_3}{2}\mathbf{t}'_3, \quad \mathbf{t}_3^- = \mathbf{t}_3 - \frac{\ell_3}{2}\mathbf{t}'_3.\end{aligned} \tag{54}$$

Since the total force is

$$\mathbf{f} = V_{\mathcal{R}}(\mathbf{t}'_1 + \mathbf{t}'_2 + \mathbf{t}'_3), \tag{55}$$

we can define a uniform bulk distribution

$$\mathbf{t}' := \mathbf{t}'_1 + \mathbf{t}'_2 + \mathbf{t}'_3 \tag{56}$$

such that the total force is $\mathbf{f}$, while the moment tensor with respect to the center p_O is zero

$$\int_{\mathcal{R}} (x - \mathsf{p}_O) \otimes \mathbf{t}'\, dV = V_{\mathcal{R}}(c - \mathsf{p}_O) \otimes \mathbf{t}' = \mathbf{O}, \tag{57}$$

since the barycenter c, which is defined by the property

$$(c - \mathsf{p}_O)V_{\mathcal{R}} = \int_{\mathcal{R}} (x - \mathsf{p}_O)\, dV\,, \tag{58}$$

coincides with p_O. Hence the force distribution made up of a uniform bulk distribution $\mathbf{t}'$ given by (56) and of a surface force distribution given by

$$\begin{aligned}\mathbf{t}_1 &= \frac{1}{V_{\mathcal{R}}}\mathbf{M}_{\mathsf{p}_O}\mathbf{e}_1,\\ \mathbf{t}_2 &= \frac{1}{V_{\mathcal{R}}}\mathbf{M}_{\mathsf{p}_O}\mathbf{e}_2,\\ \mathbf{t}_3 &= \frac{1}{V_{\mathcal{R}}}\mathbf{M}_{\mathsf{p}_O}\mathbf{e}_3,\end{aligned} \tag{59}$$

where $\mathbf{M}_{\mathsf{p}_O}$ is given by (49), belongs to the same equipowerful class as the force distribution in fig. 3.

4.4 Force distribution on the faces of a prism with triangular cross section

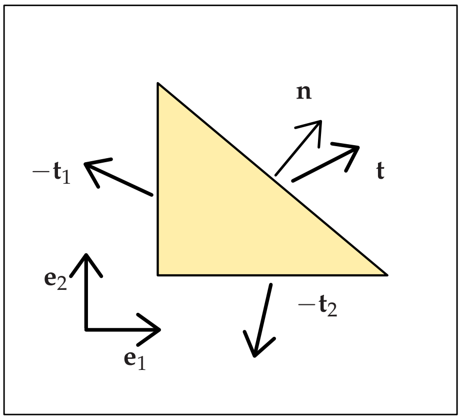

Figure 5: Force distribution on the boundary of a prism.

Let us consider a body, as in fig. 5, in the shape of a prism with triangular cross section, with edge lengths ℓ_1, ℓ_2, ℓ_3, with a uniform force distribution on the faces $\mathcal{F}_{-1}, \mathcal{F}_{-2}$ and $\mathcal{F}$, the sloping face, such that

$$A_{\mathcal{F}}\,\mathbf{t} - A_{\mathcal{F}_{-1}}\,\mathbf{t}_1 - A_{\mathcal{F}_{-2}}\,\mathbf{t}_2 = \mathbf{o}\,. \tag{60}$$

It follows

$$\ell_3\ell\,\mathbf{t} - \ell_3\ell_2\,\mathbf{t}_1 - \ell_3\ell_1\,\mathbf{t}_2 = \mathbf{o} \quad \Rightarrow \quad \mathbf{t} = \frac{1}{\ell}(\ell_2\mathbf{t}_1 + \ell_1\mathbf{t}_2), \tag{61}$$

where $\ell := \sqrt{\ell_1^2 + \ell_2^2}$. By choosing $\mathsf{p_O}$ in the sloping face center, the moment tensor is

$$\mathbf{M}_{\mathsf{p_O}} = -\frac{\ell_1}{2}\ell_2\ell_3\,\mathbf{e}_1 \otimes (-\mathbf{t}_1) - \frac{\ell_2}{2}\ell_1\ell_3\,\mathbf{e}_2 \otimes (-\mathbf{t}_2) = V_{\mathcal{R}}(\mathbf{e}_1 \otimes \mathbf{t}_1 + \mathbf{e}_2 \otimes \mathbf{t}_2) \tag{62}$$

Note that

$$\mathbf{t}_1 = \frac{1}{V_{\mathcal{R}}}\mathbf{M}_{\mathsf{p_O}}\mathbf{e}_1,$$
$$\mathbf{t}_2 = \frac{1}{V_{\mathcal{R}}}\mathbf{M}_{\mathsf{p_O}}\mathbf{e}_2, \tag{63}$$
$$\mathbf{t} = \frac{1}{V_{\mathcal{R}}}\mathbf{M}_{\mathsf{p_O}}\mathbf{n},$$

where

$$\mathbf{n} := \frac{1}{\ell}(\ell_2\mathbf{e}_1 + \ell_1\mathbf{e}_2) \tag{64}$$

is the outward unit normal to the sloping face.

4.5 Forces applied at the face centres of a parallelepiped

Let us consider a parallelepiped with edges $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and forces $\mathbf{f}_1, -\mathbf{f}_1, \mathbf{f}_2, -\mathbf{f}_2, \mathbf{f}_3, -\mathbf{f}_3$ applied respectively at the center of opposite faces, as in fig. 6. The total force is zero and the total moment

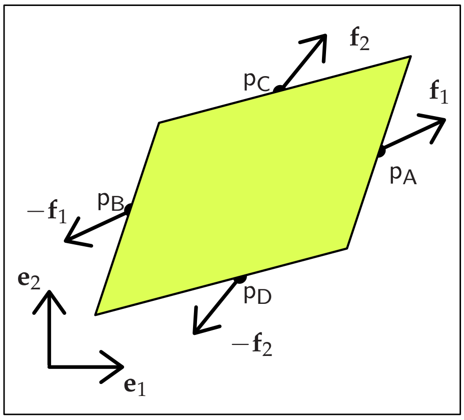

Figure 6: Forces applied at the center of the faces of a parallelpiped.

tensor is

$$\mathbf{M} = \mathbf{u}_1 \otimes \mathbf{f}_1 + \mathbf{u}_2 \otimes \mathbf{f}_2 + \mathbf{u}_3 \otimes \mathbf{f}_3. \tag{65}$$

Denoting by $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ the outward unit normal vectors, let us express $\mathbf{n}_1$ as a linear combination of the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

$$\mathbf{n}_1 = \nu_{11}\mathbf{u}_1 + \nu_{21}\mathbf{u}_2 + \nu_{31}\mathbf{u}_3. \tag{66}$$

This linear combination can be transformed, dividing by ν_{11}, into

$$\mathbf{u}_1 = h_1\mathbf{n}_1 + \alpha_{21}\mathbf{u}_2 + \alpha_{31}\mathbf{u}_3. \tag{67}$$

Since $\mathbf{n}_1$ is a unit vector orthogonal to $\mathbf{u}_2$ and $\mathbf{u}_3$ then

$$\mathbf{u}_1 \cdot \mathbf{n}_1 = (h_1\mathbf{n}_1 + \alpha_{21}\mathbf{u}_2 + \alpha_{31}\mathbf{u}_3) \cdot \mathbf{n}_1 = h_1\,. \tag{68}$$

Further, by the properties of the volume function,

$$\begin{aligned} V_{\mathcal{R}} &= \mathrm{vol}\,(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \mathrm{vol}\,(\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_1) = \mathrm{vol}\,(\mathbf{u}_2, \mathbf{u}_3, h_1\mathbf{n}_1 + \alpha_{21}\mathbf{u}_2 + \alpha_{31}\mathbf{u}_3) \\ &= \mathrm{vol}\,(\mathbf{u}_2, \mathbf{u}_3, h_1\mathbf{n}_1) = h_1\,\mathrm{vol}\,(\mathbf{u}_2, \mathbf{u}_3, \mathbf{n}_1) = h_1 A_{\mathcal{F}_1} \\ &\Rightarrow \quad h_1 = \frac{V_{\mathcal{R}}}{A_{\mathcal{F}_1}}. \end{aligned} \tag{69}$$

Notice that h_1 is nothing but the *height* of the parallelepiped with respect to the face $\mathcal{F}_1$. Summarizing:

$$\mathbf{u}_1 \cdot \mathbf{n}_1 = \frac{V_{\mathcal{R}}}{A_{\mathcal{F}_1}}\,, \quad \mathbf{u}_2 \cdot \mathbf{n}_1 = 0\,, \quad \mathbf{u}_3 \cdot \mathbf{n}_1 = 0\,. \tag{70}$$

From the expression for the moment tensor, it follows that

$$\mathbf{M}\mathbf{n}_1 = (\mathbf{u}_1 \otimes \mathbf{f}_1 + \mathbf{u}_2 \otimes \mathbf{f}_2 + \mathbf{u}_3 \otimes \mathbf{f}_3)\,\mathbf{n}_1 = (\mathbf{u}_1 \cdot \mathbf{n}_1)\mathbf{f}_1 = \frac{V_{\mathcal{R}}}{A_{\mathcal{F}_1}}\mathbf{f}_1 \tag{71}$$

and finally

$$\frac{\mathbf{M}}{V_{\mathcal{R}}}\mathbf{n}_1 = \frac{\mathbf{f}_1}{A_{\mathcal{F}_1}}. \tag{72}$$

Similar expressions can be obtained by applying $\mathbf{M}$ to $\mathbf{n}_2$ and $\mathbf{n}_3$.

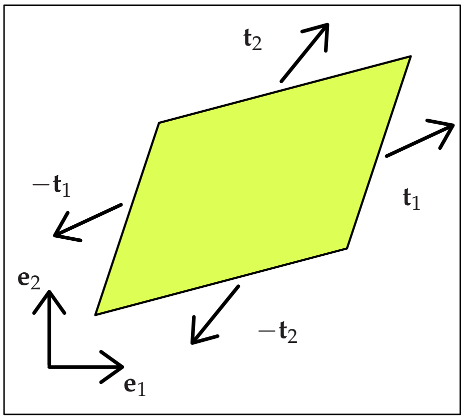

Figure 7: Uniform force distributions applied on the faces of a parallelepiped.

4.6 Force distribution on the faces of a rectangular parallelepiped

Let us consider a parallelepiped with edges $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and uniform force distributions $\mathbf{t}_1, -\mathbf{t}_1, \mathbf{t}_2, -\mathbf{t}_2, \mathbf{t}_3, -\mathbf{t}_3$ applied respectively on opposite faces, as in fig. 7. The total force is zero and the total moment tensor is

$$\begin{aligned}
\mathbf{M} &= \int_{\mathcal{F}_1} (x - \mathsf{p}_O) \otimes \mathbf{t}_1 \, dA + \int_{\mathcal{F}_{-1}} (x - \mathsf{p}_O) \otimes (-\mathbf{t}_1) \, dA \\
&\quad + \int_{\mathcal{F}_2} (x - \mathsf{p}_O) \otimes \mathbf{t}_2 \, dA + \int_{\mathcal{F}_{-2}} (x - \mathsf{p}_O) \otimes (-\mathbf{t}_2) \, dA \\
&\quad + \int_{\mathcal{F}_3} (x - \mathsf{p}_O) \otimes \mathbf{t}_3 \, dA + \int_{\mathcal{F}_{-3}} (x - \mathsf{p}_O) \otimes (-\mathbf{t}_3) \, dA \\
&= (c_{\mathcal{F}_1} - \mathsf{p}_O) \otimes \mathbf{t}_1 \, A_{\mathcal{F}_1} + (c_{\mathcal{F}_{-1}} - \mathsf{p}_O) \otimes (-\mathbf{t}_1) \, A_{\mathcal{F}_{-1}} \\
&\quad + (c_{\mathcal{F}_2} - \mathsf{p}_O) \otimes \mathbf{t}_2 \, A_{\mathcal{F}_2} + (c_{\mathcal{F}_{-2}} - \mathsf{p}_O) \otimes (-\mathbf{t}_2) \, A_{\mathcal{F}_{-2}} \\
&\quad + (c_{\mathcal{F}_3} - \mathsf{p}_O) \otimes \mathbf{t}_3 \, A_{\mathcal{F}_3} + (c_{\mathcal{F}_{-3}} - \mathsf{p}_O) \otimes (-\mathbf{t}_3) \, A_{\mathcal{F}_{-3}} \\
&= A_{\mathcal{F}_1} \mathbf{u}_1 \otimes \mathbf{t}_1 + A_{\mathcal{F}_2} \mathbf{u}_2 \otimes \mathbf{t}_2 + A_{\mathcal{F}_3} \mathbf{u}_3 \otimes \mathbf{t}_3
\end{aligned} \tag{73}$$

where we have set

$$\begin{aligned}
\int_{\mathcal{F}_1} dA &= A_{\mathcal{F}_1} = \mathrm{vol}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{n}_1)\,, \\
\int_{\mathcal{F}_2} dA &= A_{\mathcal{F}_2} = \mathrm{vol}(\mathbf{u}_3, \mathbf{u}_1, \mathbf{n}_2)\,, \\
\int_{\mathcal{F}_3} dA &= A_{\mathcal{F}_3} = \mathrm{vol}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}_3)\,,
\end{aligned} \tag{74}$$

with $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ the outward unit normal vectors to the faces $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. By using (70) we get

$$\mathbf{M}\,\mathbf{n}_1 = (A_{\mathcal{F}_1} \mathbf{u}_1 \otimes \mathbf{t}_1 + A_{\mathcal{F}_2} \mathbf{u}_2 \otimes \mathbf{t}_2 + A_{\mathcal{F}_3} \mathbf{u}_3 \otimes \mathbf{t}_3)\mathbf{n}_1 = (\mathbf{u}_1 \cdot \mathbf{n}_1)\, A_{\mathcal{F}_1} \mathbf{t}_1 = V_{\mathcal{R}} \mathbf{t}_1 \tag{75}$$

and finally

$$\frac{\mathbf{M}}{V_{\mathcal{R}}} \mathbf{n}_1 = \mathbf{t}_1 . \tag{76}$$

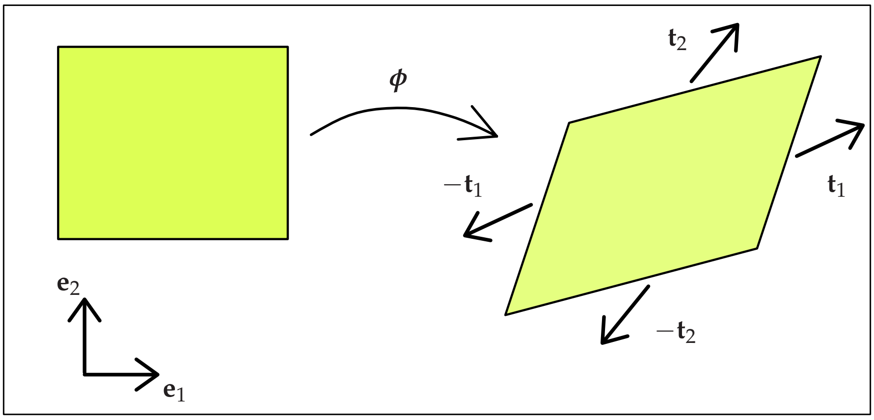

Figure 8: Uniform force distributions applied on the faces of a deformed parallelepiped.

Similar expressions can be obtained by applying $\mathbf{M}$ to $\mathbf{n}_2$ and $\mathbf{n}_3$. It is worth noticing that vectors $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ in general are non orthogonal, like the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

4.7 Force distribution on the faces of a deformed parallelepiped

Let us consider the parallelpiped with edges $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of the previous paragraph, this time seen as resulting from applying an affine deformation with gradient $\mathbf{F}$ to a parallelepiped with edges $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3\}$, as in fig. 8. The total force is again zero and the total moment tensor (73) can be transformed in the following way

$$\begin{aligned}\mathbf{M} &= A_{\mathcal{F}_1}\,\mathbf{u}_1 \otimes \mathbf{t}_1 + A_{\mathcal{F}_2}\,\mathbf{u}_2 \otimes \mathbf{t}_2 + A_{\mathcal{F}_3}\,\mathbf{u}_3 \otimes \mathbf{t}_3 \\ &= A_{\mathcal{F}_1}\,\mathbf{F}\bar{\mathbf{u}}_1 \otimes \mathbf{t}_1 + A_{\mathcal{F}_2}\,\mathbf{F}\bar{\mathbf{u}}_2 \otimes \mathbf{t}_2 + A_{\mathcal{F}_3}\,\mathbf{F}\bar{\mathbf{u}}_3 \otimes \mathbf{t}_3 \\ &= A_{\mathcal{F}_1}\,(\bar{\mathbf{u}}_1 \otimes \mathbf{t}_1)\mathbf{F}^{\mathsf{T}} + A_{\mathcal{F}_2}\,(\bar{\mathbf{u}}_2 \otimes \mathbf{t}_2)\mathbf{F}^{\mathsf{T}} + A_{\mathcal{F}_3}\,(\bar{\mathbf{u}}_3 \otimes \mathbf{t}_3)\mathbf{F}^{\mathsf{T}}\end{aligned} \tag{77}$$

Denoting by $\{\bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2, \bar{\mathbf{n}}_3\}$ the outward unit normal vector to the faces $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}_2$ and $\bar{\mathcal{F}}_3$ of the parallelpiped $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3\}$, let us consider the linear combination

$$\bar{\mathbf{u}}_1 = \bar{h}_1\bar{\mathbf{n}}_1 + \bar{\alpha}_{21}\bar{\mathbf{u}}_2 + \bar{\alpha}_{31}\bar{\mathbf{u}}_3. \tag{78}$$

Since $\bar{\mathbf{n}}_1$ is a unit vector orthogonal to $\bar{\mathbf{u}}_2$ and $\bar{\mathbf{u}}_3$, we get

$$\bar{\mathbf{u}}_1 \cdot \bar{\mathbf{n}}_1 = (\bar{h}_1\bar{\mathbf{n}}_1 + \bar{\alpha}_{21}\bar{\mathbf{u}}_2 + \bar{\alpha}_{31}\bar{\mathbf{u}}_3) \cdot \bar{\mathbf{n}}_1 = \bar{h}_1\,. \tag{79}$$

Further, by the properties of the volume function,

$$\begin{aligned}V_{\bar{\mathcal{R}}} &= \mathrm{vol}\,(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3) = \mathrm{vol}\,(\bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_1) = \mathrm{vol}\,(\bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{h}_1\bar{\mathbf{n}}_1 + \bar{\alpha}_{21}\bar{\mathbf{u}}_2 + \bar{\alpha}_{31}\bar{\mathbf{u}}_3) \\ &= \mathrm{vol}\,(\bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{h}_1\bar{\mathbf{n}}_1) = \bar{h}_1\,\mathrm{vol}\,(\bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{n}}_1) = \bar{h}_1 A_{\bar{\mathcal{F}}_1} \\ \Rightarrow \quad & \bar{h}_1 = \frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}}.\end{aligned} \tag{80}$$

Summarizing:

$$\bar{\mathbf{u}}_1 \cdot \bar{\mathbf{n}}_1 = \frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}}\,, \quad \bar{\mathbf{u}}_2 \cdot \bar{\mathbf{n}}_1 = 0\,, \quad \bar{\mathbf{u}}_3 \cdot \bar{\mathbf{n}}_1 = 0\,. \tag{81}$$

Transforming the previous expressions into

$$\bar{\mathbf{n}}_1 \cdot \mathbf{F}^{-1}\mathbf{u}_1 = \frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}}, \quad \bar{\mathbf{n}}_1 \cdot \mathbf{F}^{-1}\mathbf{u}_2 = 0, \quad \bar{\mathbf{n}}_1 \cdot \mathbf{F}^{-1}\mathbf{u}_3 = 0, \tag{82}$$

$$(\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1 \cdot \mathbf{u}_1 = \frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}}, \quad (\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1 \cdot \mathbf{u}_2 = 0, \quad (\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1 \cdot \mathbf{u}_3 = 0, \tag{83}$$

we can notice that the vector $(\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1$ is orthogonal to both $\mathbf{u}_2$ and $\mathbf{u}_3$, like $\mathbf{n}_1$. Hence we can set

$$(\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1 = \tilde{k}_1\mathbf{n}_1. \tag{84}$$

It follows that

$$(\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1 \cdot \mathbf{u}_1 = \tilde{k}_1\mathbf{n}_1 \cdot \mathbf{u}_1 \tag{85}$$

and, by (83) and (70),

$$\frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}} = \tilde{k}_1 \frac{V_{\mathcal{R}}}{A_{\mathcal{F}_1}}. \tag{86}$$

Thus it is

$$\tilde{k}_1 = \frac{V_{\bar{\mathcal{R}}}}{V_{\mathcal{R}}}\frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}} = \frac{1}{\det \mathbf{F}}\frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}}. \tag{87}$$

Setting

$$k_1 := \frac{A_{\mathcal{F}_1}}{A_{\bar{\mathcal{F}}_1}} \tag{88}$$

the expression (84) can be rewritten

$$(\det \mathbf{F})(\mathbf{F}^{-1})^{\mathsf{T}}\bar{\mathbf{n}}_1 = k_1\mathbf{n}_1. \tag{89}$$

We call *cofactor* of $\mathbf{F}$ the tensor

$$\operatorname{cof}\mathbf{F} := (\det \mathbf{F})(\mathbf{F}^{-1})^{\mathsf{T}}. \tag{90}$$

Since $\mathbf{n}_1$ is a unit vector then the ratio between the areas can be computed by the formula

$$k_1 = \|(\operatorname{cof}\mathbf{F})\,\bar{\mathbf{n}}_1\|, \tag{91}$$

while the unit normal vector $\mathbf{n}_1$ can be computed as

$$\mathbf{n}_1 = \frac{\operatorname{cof}\mathbf{F}}{\|(\operatorname{cof}\mathbf{F})\,\bar{\mathbf{n}}_1\|}\,\bar{\mathbf{n}}_1. \tag{92}$$

The moment tensor expression (77) can be transformed into the following

$$\mathbf{M}(\mathbf{F}^{\mathsf{T}})^{-1} = A_{\mathcal{F}_1}(\bar{\mathbf{u}}_1 \otimes \mathbf{t}_1) + A_{\mathcal{F}_2}(\bar{\mathbf{u}}_2 \otimes \mathbf{t}_2) + A_{\mathcal{F}_3}(\bar{\mathbf{u}}_3 \otimes \mathbf{t}_3). \tag{93}$$

If we apply this tensor to the unit normal vector $\bar{\mathbf{n}}_1$ we get

$$\begin{aligned}\mathbf{M}(\mathbf{F}^{\mathsf{T}})^{-1}\bar{\mathbf{n}}_1 &= \big(A_{\mathcal{F}_1}(\bar{\mathbf{u}}_1 \otimes \mathbf{t}_1) + A_{\mathcal{F}_2}(\bar{\mathbf{u}}_2 \otimes \mathbf{t}_2) + A_{\mathcal{F}_3}(\bar{\mathbf{u}}_3 \otimes \mathbf{t}_3)\big)\bar{\mathbf{n}}_1\\ &= A_{\mathcal{F}_1}(\bar{\mathbf{u}}_1 \cdot \bar{\mathbf{n}}_1)\mathbf{t}_1 = A_{\mathcal{F}_1}\frac{V_{\bar{\mathcal{R}}}}{A_{\bar{\mathcal{F}}_1}}\mathbf{t}_1\end{aligned} \tag{94}$$

from which it follows

$$\frac{\mathbf{M}}{V_{\bar{\mathcal{R}}}}(\mathbf{F}^{\mathsf{T}})^{-1}\bar{\mathbf{n}}_1 = k_1\mathbf{t}_1 \tag{95}$$

and also

$$\frac{\mathbf{M}}{V_{\mathcal{R}}}(\det \mathbf{F})(\mathbf{F}^{\mathsf{T}})^{-1}\bar{\mathbf{n}}_1 = \frac{\mathbf{M}}{V_{\mathcal{R}}}(\operatorname{cof} \mathbf{F})\,\bar{\mathbf{n}}_1 = k_1 \mathbf{t}_1. \tag{96}$$

The traction

$$\bar{\mathbf{t}}_1 := k_1 \mathbf{t}_1 \tag{97}$$

is charcterized, through (88), by the following property

$$\bar{\mathbf{t}}_1 A_{\bar{\mathcal{F}}_1} = \mathbf{t}_1 A_{\mathcal{F}_1}. \tag{98}$$

Similar expressions can be obtained for any other face. Looking at $\bar{\mathbf{t}}_1$, $\bar{\mathbf{t}}_2$, $\bar{\mathbf{t}}_3$ as force distributions on the faces of the parallelpiped with edges $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3\}$, their moment tensor turns out to be

$$\begin{aligned}\bar{\mathbf{M}} &= A_{\bar{\mathcal{F}}_1}(\bar{\mathbf{u}}_1 \otimes \bar{\mathbf{t}}_1) + A_{\bar{\mathcal{F}}_2}(\bar{\mathbf{u}}_2 \otimes \bar{\mathbf{t}}_2) + A_{\bar{\mathcal{F}}_3}(\bar{\mathbf{u}}_3 \otimes \bar{\mathbf{t}}_3)\\ &= k_1 A_{\bar{\mathcal{F}}_1}(\bar{\mathbf{u}}_1 \otimes \mathbf{t}_1) + k_2 A_{\bar{\mathcal{F}}_2}(\bar{\mathbf{u}}_2 \otimes \mathbf{t}_2) + k_3 A_{\bar{\mathcal{F}}_3}(\bar{\mathbf{u}}_3 \otimes \mathbf{t}_3)\\ &= A_{\mathcal{F}_1}(\bar{\mathbf{u}}_1 \otimes \mathbf{t}_1) + A_{\mathcal{F}_2}(\bar{\mathbf{u}}_2 \otimes \mathbf{t}_2) + A_{\mathcal{F}_3}(\bar{\mathbf{u}}_3 \otimes \mathbf{t}_3)\\ &= \mathbf{M}(\mathbf{F}^{\mathsf{T}})^{-1}\end{aligned} \tag{99}$$

Hence (95) can be transformed into

$$\frac{\bar{\mathbf{M}}}{V_{\bar{\mathcal{R}}}}\bar{\mathbf{n}}_1 = \bar{\mathbf{t}}_1. \tag{100}$$

4.8 Cofactor matrix

In order to write an expression for the cofactor matrix defined in (90), let us replace the vectors $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3\}$ with the vectors of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Let $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ be the unit normal vectors to the faces of the parallelepiped generated by the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ obtained by applying to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the same deformation as in fig. 8. Notice that the area of $\mathcal{F}_1$, already defined in (70), can be given the expression

$$A_{\mathcal{F}_1} = \operatorname{vol}(\mathbf{u}_2, \mathbf{u}_3, \mathbf{n}_1) = \operatorname{vol}(\mathbf{n}_1, \mathbf{u}_2, \mathbf{u}_3) = \operatorname{vol}(\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3)\,. \tag{101}$$

The basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be expressed as linear combinations of the independent vectors $\{\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3\}$, as follows

$$\begin{aligned}\mathbf{e}_1 &= \gamma_{11}\mathbf{n}_1 + \gamma_{21}\mathbf{F}\mathbf{e}_2 + \gamma_{31}\mathbf{F}\mathbf{e}_3\,,\\ \mathbf{e}_2 &= \gamma_{12}\mathbf{n}_1 + \gamma_{22}\mathbf{F}\mathbf{e}_2 + \gamma_{32}\mathbf{F}\mathbf{e}_3\,,\\ \mathbf{e}_3 &= \gamma_{13}\mathbf{n}_1 + \gamma_{23}\mathbf{F}\mathbf{e}_2 + \gamma_{33}\mathbf{F}\mathbf{e}_3\,.\end{aligned} \tag{102}$$

Since $\mathbf{n}_1$ is orthogonal to both $\mathbf{F}\mathbf{e}_2$ and $\mathbf{F}\mathbf{e}_3$, it turns out

$$\begin{aligned}\mathbf{n}_1 \cdot \mathbf{e}_1 &= \gamma_{11}\,,\\ \mathbf{n}_1 \cdot \mathbf{e}_2 &= \gamma_{12}\,,\\ \mathbf{n}_1 \cdot \mathbf{e}_3 &= \gamma_{13}\,.\end{aligned} \tag{103}$$

Hence

$$\begin{aligned}\operatorname{vol}(\mathbf{e}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) &= \operatorname{vol}(\gamma_{11}\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = (\mathbf{n}_1 \cdot \mathbf{e}_1)\operatorname{vol}(\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = (\mathbf{n}_1 \cdot \mathbf{e}_1) A_{\mathcal{F}1}\,,\\ \operatorname{vol}(\mathbf{e}_2, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) &= \operatorname{vol}(\gamma_{12}\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = (\mathbf{n}_1 \cdot \mathbf{e}_2)\operatorname{vol}(\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = (\mathbf{n}_1 \cdot \mathbf{e}_2) A_{\mathcal{F}1}\,,\\ \operatorname{vol}(\mathbf{e}_3, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) &= \operatorname{vol}(\gamma_{13}\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = (\mathbf{n}_1 \cdot \mathbf{e}_3)\operatorname{vol}(\mathbf{n}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = (\mathbf{n}_1 \cdot \mathbf{e}_3) A_{\mathcal{F}1}\,.\end{aligned} \tag{104}$$

From (89) and (88), since $A_{\bar{\mathcal{F}}_1} = 1$, we get

$$\operatorname{cof}\mathbf{F}\,\mathbf{e}_1 = k_1\mathbf{n}_1 = A_{\mathcal{F}_1}\mathbf{n}_1. \tag{105}$$

We can compute the components of this vector, which will be arranged in the first column of the matrix of cof **F**, by using the scalar product with the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ thus obtaining, through (104),

$$\begin{aligned}
\operatorname{cof}\mathbf{F}\,\mathbf{e}_1\cdot\mathbf{e}_1 &= A_{\mathcal{F}_1}(\mathbf{n}_1\cdot\mathbf{e}_1) = \operatorname{vol}(\mathbf{e}_1, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = f_{22}f_{33} - f_{23}f_{32}\,,\\
\operatorname{cof}\mathbf{F}\,\mathbf{e}_1\cdot\mathbf{e}_2 &= A_{\mathcal{F}_1}(\mathbf{n}_1\cdot\mathbf{e}_2) = \operatorname{vol}(\mathbf{e}_2, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = f_{32}f_{13} - f_{33}f_{12}\,,\\
\operatorname{cof}\mathbf{F}\,\mathbf{e}_1\cdot\mathbf{e}_3 &= A_{\mathcal{F}_1}(\mathbf{n}_1\cdot\mathbf{e}_3) = \operatorname{vol}(\mathbf{e}_3, \mathbf{F}\mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = f_{12}f_{23} - f_{13}f_{22}\,.
\end{aligned} \tag{106}$$

From the area of $\mathcal{F}_2$ we get

$$\begin{aligned}
\operatorname{cof}\mathbf{F}\,\mathbf{e}_2\cdot\mathbf{e}_1 &= \operatorname{vol}(\mathbf{F}\mathbf{e}_1, \mathbf{e}_1, \mathbf{F}\mathbf{e}_3) = f_{23}f_{31} - f_{21}f_{33}\,,\\
\operatorname{cof}\mathbf{F}\,\mathbf{e}_2\cdot\mathbf{e}_2 &= \operatorname{vol}(\mathbf{F}\mathbf{e}_1, \mathbf{e}_2, \mathbf{F}\mathbf{e}_3) = f_{33}f_{11} - f_{31}f_{13}\,,\\
\operatorname{cof}\mathbf{F}\,\mathbf{e}_2\cdot\mathbf{e}_3 &= \operatorname{vol}(\mathbf{F}\mathbf{e}_1, \mathbf{e}_3, \mathbf{F}\mathbf{e}_3) = f_{13}f_{21} - f_{11}f_{23}\,,
\end{aligned} \tag{107}$$

and, from the area of $\mathcal{F}_3$,

$$\begin{aligned}
\operatorname{cof}\mathbf{F}\,\mathbf{e}_3\cdot\mathbf{e}_1 &= \operatorname{vol}(\mathbf{F}\mathbf{e}_1, \mathbf{F}\mathbf{e}_2, \mathbf{e}_1) = f_{21}f_{32} - f_{22}f_{31}\,,\\
\operatorname{cof}\mathbf{F}\,\mathbf{e}_3\cdot\mathbf{e}_2 &= \operatorname{vol}(\mathbf{F}\mathbf{e}_1, \mathbf{F}\mathbf{e}_2, \mathbf{e}_2) = f_{31}f_{12} - f_{32}f_{11}\,,\\
\operatorname{cof}\mathbf{F}\,\mathbf{e}_3\cdot\mathbf{e}_3 &= \operatorname{vol}(\mathbf{F}\mathbf{e}_1, \mathbf{F}\mathbf{e}_2, \mathbf{e}_3) = f_{11}f_{22} - f_{12}f_{21}\,.
\end{aligned} \tag{108}$$

Hence the matrix of the cofactor of **F** turns out to be

$$[\operatorname{cof}\mathbf{F}] = \begin{pmatrix} f_{22}f_{33} - f_{23}f_{32} & f_{23}f_{31} - f_{21}f_{33} & f_{21}f_{32} - f_{22}f_{31}\\ f_{32}f_{13} - f_{33}f_{12} & f_{33}f_{11} - f_{31}f_{13} & f_{31}f_{12} - f_{32}f_{11}\\ f_{12}f_{23} - f_{13}f_{22} & f_{13}f_{21} - f_{11}f_{23} & f_{11}f_{22} - f_{12}f_{21} \end{pmatrix}. \tag{109}$$

For a *plane deformation*, where the matrix of **F** is

$$[\mathbf{F}] = \begin{pmatrix} f_{11} & f_{12} & 0\\ f_{21} & f_{22} & 0\\ 0 & 0 & 1 \end{pmatrix}, \tag{110}$$

the cofactor matrix becomes

$$[\operatorname{cof}\mathbf{F}] = \begin{pmatrix} f_{22} & -f_{21} & 0\\ -f_{12} & f_{11} & 0\\ 0 & 0 & f_{11}f_{22} - f_{12}f_{21} \end{pmatrix}. \tag{111}$$

4.9 Force distribution on the faces of a tetrahedron

Let us consider uniform force distributions over the faces of a tetrahedron, whose edges are $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, as in fig. 9. The volume of a tetrahedron inscribed in a parallelepiped is (see fig. 10)

$$V_{\mathcal{R}} = \frac{1}{6}\operatorname{vol}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \tag{112}$$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are independent vectors we can write

$$\mathbf{b}_1 = -\mathbf{u}_2 + \mathbf{u}_3, \quad \mathbf{b}_2 = -\mathbf{u}_3 + \mathbf{u}_1, \quad \mathbf{b}_3 = -\mathbf{u}_1 + \mathbf{u}_2, \tag{113}$$

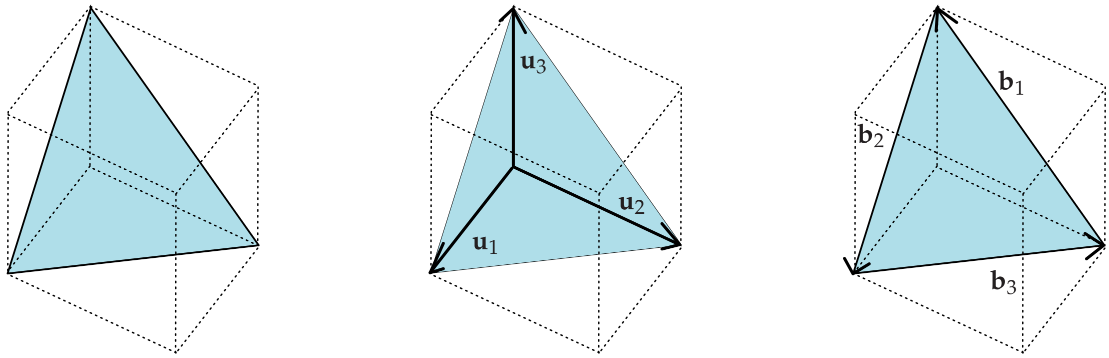

Figure 9: Body in the shape of a tetrahedron.

from which it follows

$$\begin{aligned}
\mathrm{vol}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{u}_3) &= \mathrm{vol}(-\mathbf{u}_2+\mathbf{u}_3,-\mathbf{u}_3+\mathbf{u}_1,\mathbf{u}_3) = \mathrm{vol}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = 6V_{\mathcal{R}},\\
\mathrm{vol}(\mathbf{b}_2,\mathbf{b}_3,\mathbf{u}_1) &= \mathrm{vol}(-\mathbf{u}_3+\mathbf{u}_1,-\mathbf{u}_1+\mathbf{u}_2,\mathbf{u}_1) = \mathrm{vol}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = 6V_{\mathcal{R}},\\
\mathrm{vol}(\mathbf{b}_3,\mathbf{b}_1,\mathbf{u}_2) &= \mathrm{vol}(-\mathbf{u}_1+\mathbf{u}_2,-\mathbf{u}_2+\mathbf{u}_3,\mathbf{u}_2) = \mathrm{vol}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3) = 6V_{\mathcal{R}}.
\end{aligned} \tag{114}$$

In turn, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ can be written as linear combinations of any two of the three vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, together with the outward unit normal vector to the sloping face $\mathcal{F}$

$$\begin{aligned}
\mathbf{u}_1 &= \beta_1\mathbf{n} + \beta_{21}\,\mathbf{b}_2 + \beta_{31}\,\mathbf{b}_3,\\
\mathbf{u}_2 &= \beta_2\mathbf{n} + \beta_{32}\,\mathbf{b}_3 + \beta_{12}\,\mathbf{b}_1,\\
\mathbf{u}_3 &= \beta_3\mathbf{n} + \beta_{13}\,\mathbf{b}_1 + \beta_{23}\,\mathbf{b}_2.
\end{aligned} \tag{115}$$

If we rewrite (114) by using the expressions above we get

$$\begin{aligned}
\mathrm{vol}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{u}_3) &= \mathrm{vol}(\mathbf{b}_1,\mathbf{b}_2,\beta_3\mathbf{n}+\beta_{13}\,\mathbf{b}_1+\beta_{23}\,\mathbf{b}_2) = \beta_3\,\mathrm{vol}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{n}) = 2\beta_3\,A_{\mathcal{F}}\,,\\
\mathrm{vol}(\mathbf{b}_2,\mathbf{b}_3,\mathbf{u}_1) &= \mathrm{vol}(\mathbf{b}_2,\mathbf{b}_3,\beta_1\mathbf{n}+\beta_{21}\,\mathbf{b}_2+\beta_{31}\,\mathbf{b}_3) = \beta_1\,\mathrm{vol}(\mathbf{b}_2,\mathbf{b}_3,\mathbf{n}) = 2\beta_1\,A_{\mathcal{F}}\,,\\
\mathrm{vol}(\mathbf{b}_3,\mathbf{b}_1,\mathbf{u}_2) &= \mathrm{vol}(\mathbf{b}_3,\mathbf{b}_1,\beta_2\mathbf{n}+\beta_{32}\,\mathbf{b}_3+\beta_{12}\,\mathbf{b}_1) = \beta_2\,\mathrm{vol}(\mathbf{b}_3,\mathbf{b}_1,\mathbf{n}) = 2\beta_2\,A_{\mathcal{F}}\,,
\end{aligned} \tag{116}$$

where the area of the sloping face is given by

$$A_{\mathcal{F}} = \frac{1}{2}\,\mathrm{vol}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{n}) = \frac{1}{2}\,\mathrm{vol}(\mathbf{b}_2,\mathbf{b}_3,\mathbf{n}) = \frac{1}{2}\,\mathrm{vol}(\mathbf{b}_3,\mathbf{b}_1,\mathbf{n}). \tag{117}$$

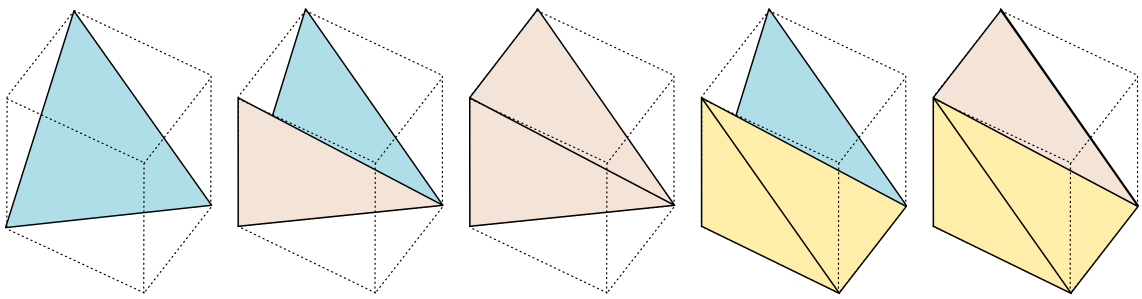

Figure 10: How half a parallelpiped can be split into three different tetrahedrons (blue, pink, yellow) of equal volume.

By using (113), it can be checked that the three expressions have the same value

$$\begin{aligned}
\mathrm{vol}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{n}) &= \mathrm{vol}(\mathbf{u}_3-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_3,\mathbf{n}) = \mathrm{vol}(\mathbf{u}_3,\mathbf{u}_1,\mathbf{n}) + \mathrm{vol}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{n}) + \mathrm{vol}(\mathbf{u}_2,\mathbf{u}_3,\mathbf{n})\,,\\
\mathrm{vol}(\mathbf{b}_2,\mathbf{b}_3,\mathbf{n}) &= \mathrm{vol}(\mathbf{u}_1-\mathbf{u}_3,\mathbf{u}_2-\mathbf{u}_1,\mathbf{n}) = \mathrm{vol}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{n}) + \mathrm{vol}(\mathbf{u}_2,\mathbf{u}_3,\mathbf{n}) + \mathrm{vol}(\mathbf{u}_3,\mathbf{u}_1,\mathbf{n})\,,\\
\mathrm{vol}(\mathbf{b}_3,\mathbf{b}_1,\mathbf{n}) &= \mathrm{vol}(\mathbf{u}_2-\mathbf{u}_1,\mathbf{u}_3-\mathbf{u}_2,\mathbf{n}) = \mathrm{vol}(\mathbf{u}_2,\mathbf{u}_3,\mathbf{n}) + \mathrm{vol}(\mathbf{u}_3,\mathbf{u}_1,\mathbf{n}) + \mathrm{vol}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{n})\,.
\end{aligned} \tag{118}$$

Comparing (116) and (114) we find that $\beta_1 = \beta_2 = \beta_3$ and that their common value is

$$h = 3\frac{V_{\mathcal{R}}}{A_{\mathcal{F}}} \tag{119}$$

which is nothing but the height of the tetrahedron as a pyramid with base $\mathcal{F}$.

Let us name the other three faces $\mathcal{F}_{-1}$, $\mathcal{F}_{-2}$, $\mathcal{F}_{-3}$ and let $-\mathbf{t}_1$, $-\mathbf{t}_2$, $-\mathbf{t}_3$ be the corresponding uniform force distributions. The uniform force distribution $\mathbf{t}$ on the sloping face $\mathcal{F}$ is assumed to be such that

$$A_{\mathcal{F}}\mathbf{t} - A_{\mathcal{F}_{-1}}\mathbf{t}_1 - A_{\mathcal{F}_{-2}}\mathbf{t}_2 - A_{\mathcal{F}_{-3}}\mathbf{t}_3 = \mathbf{o}. \tag{120}$$

The moment tensor with respect to p_O is

$$\begin{aligned}
\mathbf{M}_{\mathsf{p}_O} &= (c_{\mathcal{F}} - \mathsf{p}_O) \otimes (\mathbf{t}A_{\mathcal{F}})\\
&+ (c_{\mathcal{F}_{-1}} - \mathsf{p}_O) \otimes (-\mathbf{t}_1 A_{\mathcal{F}_{-1}})\\
&+ (c_{\mathcal{F}_{-2}} - \mathsf{p}_O) \otimes (-\mathbf{t}_2 A_{\mathcal{F}_{-2}})\\
&+ (c_{\mathcal{F}_{-3}} - \mathsf{p}_O) \otimes (-\mathbf{t}_3 A_{\mathcal{F}_{-3}})\,.
\end{aligned} \tag{121}$$

By replacing the positions of the barycenters

$$\begin{aligned}
c_{\mathcal{F}} &= \mathsf{p}_O + \mathbf{u}_1 + \frac{1}{3}(\mathbf{b}_3 - \mathbf{b}_2) = \mathsf{p}_O + \frac{1}{3}(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3)\,,\\
c_{\mathcal{F}_1} &= \mathsf{p}_O + \frac{1}{3}(\mathbf{u}_2 + \mathbf{u}_3)\,,\\
c_{\mathcal{F}_2} &= \mathsf{p}_O + \frac{1}{3}(\mathbf{u}_3 + \mathbf{u}_1)\,,\\
c_{\mathcal{F}_3} &= \mathsf{p}_O + \frac{1}{3}(\mathbf{u}_1 + \mathbf{u}_2)\,.
\end{aligned} \tag{122}$$

we get

$$\begin{aligned}
\mathbf{M}_{\mathsf{p}_O} &= \frac{1}{3}A_{\mathcal{F}}(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) \otimes \mathbf{t}\\
&- \frac{1}{3}A_{\mathcal{F}_{-1}}(\mathbf{u}_2 + \mathbf{u}_3) \otimes \mathbf{t}_1\\
&- \frac{1}{3}A_{\mathcal{F}_{-2}}(\mathbf{u}_3 + \mathbf{u}_1) \otimes \mathbf{t}_2\\
&- \frac{1}{3}A_{\mathcal{F}_{-3}}(\mathbf{u}_1 + \mathbf{u}_2) \otimes \mathbf{t}_3\,.
\end{aligned} \tag{123}$$

Again, by applying $\mathbf{M}_{\mathsf{p}_O}/V_{\mathcal{R}}$ to each outward unit normal vector we obtain the uniform force distribution on the corresponding face. This can be shown for the sloping face $\mathcal{F}$ by noticing that, because of (115) and (119),

$$\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n} = \mathbf{u}_3 \cdot \mathbf{n} = h = 3\frac{V_{\mathcal{R}}}{A_{\mathcal{F}}}\,. \tag{124}$$

Hence

$$
\begin{aligned}
\frac{1}{V_{\mathcal{R}}}\mathbf{M}_{p_O}\mathbf{n} &= \frac{A_{\mathcal{F}}}{3V_{\mathcal{R}}}((\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3)\cdot\mathbf{n})\mathbf{t}\\
&- \frac{A_{\mathcal{F}_{-1}}}{3V_{\mathcal{R}}}((\mathbf{u}_2+\mathbf{u}_3)\cdot\mathbf{n})\mathbf{t}_1\\
&- \frac{A_{\mathcal{F}_{-2}}}{3V_{\mathcal{R}}}((\mathbf{u}_3+\mathbf{u}_1)\cdot\mathbf{n})\mathbf{t}_2\\
&- \frac{A_{\mathcal{F}_{-3}}}{3V_{\mathcal{R}}}((\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{n})\mathbf{t}_3\\
&= 3\,\mathbf{t} - 2\frac{A_{\mathcal{F}_{-1}}}{A_{\mathcal{F}}}\mathbf{t}_1 - 2\frac{A_{\mathcal{F}_{-2}}}{A_{\mathcal{F}}}\mathbf{t}_2 - 2\frac{A_{\mathcal{F}_{-3}}}{A_{\mathcal{F}}}\mathbf{t}_3
\end{aligned}
\tag{125}
$$

which, by (120), becomes

$$
\frac{1}{V_{\mathcal{R}}}\mathbf{M}_{p_O}\mathbf{n} = \mathbf{t}\,. \tag{126}
$$

Denoting by $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ the opposite of the outward unit normal vectors to faces $\mathcal{F}_{-1}$, $\mathcal{F}_{-2}$, $\mathcal{F}_{-3}$, respectively, it can also be shown that

$$
\begin{aligned}
\mathbf{u}_1\cdot\mathbf{n}_1 &= 3\frac{V_{\mathcal{R}}}{A_{\mathcal{F}_{-1}}}\,, \quad \mathbf{u}_2\cdot\mathbf{n}_1 = 0\,, \quad \mathbf{u}_3\cdot\mathbf{n}_1 = 0\,,\\
\mathbf{u}_2\cdot\mathbf{n}_2 &= 3\frac{V_{\mathcal{R}}}{A_{\mathcal{F}_{-2}}}\,, \quad \mathbf{u}_3\cdot\mathbf{n}_2 = 0\,, \quad \mathbf{u}_1\cdot\mathbf{n}_2 = 0\,,\\
\mathbf{u}_3\cdot\mathbf{n}_3 &= 3\frac{V_{\mathcal{R}}}{A_{\mathcal{F}_{-3}}}\,, \quad \mathbf{u}_3\cdot\mathbf{n}_3 = 0\,, \quad \mathbf{u}_1\cdot\mathbf{n}_3 = 0\,.
\end{aligned}
\tag{127}
$$

as it was in (70), where $V_{\mathcal{R}}$ was the volume of the whole parallelepiped and $A_{\mathcal{F}_1}$ was the area of a parallelogram. If we write $\mathbf{n}$ as a linear combination of the other unit normal vectors

$$
\mathbf{n} = \nu_1\mathbf{n}_1 + \nu_2\mathbf{n}_2 + \nu_3\mathbf{n}_3\,, \tag{128}
$$

and compute the scalar product by $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ we get

$$
\nu_1 = \frac{A_{\mathcal{F}_{-1}}}{A_{\mathcal{F}}}\,, \quad \nu_2 = \frac{A_{\mathcal{F}_{-2}}}{A_{\mathcal{F}}}\,, \quad \nu_3 = \frac{A_{\mathcal{F}_{-3}}}{A_{\mathcal{F}}}\,. \tag{129}
$$

5 Symmetric and skew symmetric moment tensors

Let us consider an equipowerful class characterized by a zero total force and a symmetric moment tensor. A force distribution belonging to such a class can be chosen to be a surface distribution on the boundary of a body in the shape of a cube with volume ℓ^3. Denoting the moment tensor matrix by

$$
[\mathbf{M}] = \ell^3 \begin{pmatrix} t_{11} & t_{21} & t_{31}\\ t_{21} & t_{22} & t_{32}\\ t_{31} & t_{32} & t_{33}\end{pmatrix}, \tag{130}
$$

we get for each face

$$
\begin{aligned}
\mathbf{t}_1 &= t_{11}\mathbf{e}_1 + t_{21}\mathbf{e}_2 + t_{31}\mathbf{e}_3,\\
\mathbf{t}_2 &= t_{21}\mathbf{e}_1 + t_{22}\mathbf{e}_2 + t_{32}\mathbf{e}_3,\\
\mathbf{t}_3 &= t_{31}\mathbf{e}_1 + t_{32}\mathbf{e}_2 + t_{33}\mathbf{e}_3.
\end{aligned}
\tag{131}
$$

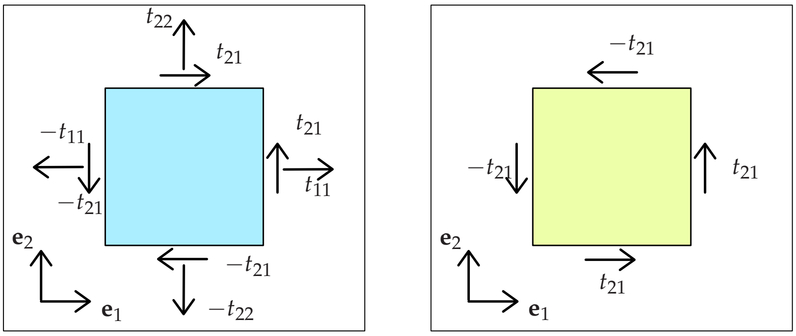

Figure 11: Force distribution with a symmetric (left) or skew symmetric (right) moment tensor.

Let us consider an equipowerful class characterized by a zero total force and a skew symmetric moment tensor. Denoting the moment tensor matrix by

$$[\mathbf{M}_{p_O}] = \ell^3 \begin{pmatrix} 0 & -t_{21} & t_{13} \\ t_{21} & 0 & -t_{32} \\ -t_{13} & t_{32} & 0 \end{pmatrix}, \tag{132}$$

we get for each face

$$\begin{aligned} \mathbf{t}_1 &= t_{21}\mathbf{e}_2 - t_{13}\mathbf{e}_3, \\ \mathbf{t}_2 &= t_{32}\mathbf{e}_3 - t_{21}\mathbf{e}_1, \\ \mathbf{t}_3 &= t_{13}\mathbf{e}_1 - t_{32}\mathbf{e}_2. \end{aligned} \tag{133}$$

The two force distributions are usually depicted as shown in fig. 11.