Linear elasticity for affine bodies

Contents

1 Small deformations

Very often bodies deform very little. That is why it is useful to derive the balance equations and the material response for “small deformations”. Let us consider a trajectory generated by affine deformations depending on a control parameter β

$$\phi_\beta(\bar{\mathsf{p}}_\mathsf{A}) = \phi_\beta(\bar{\mathsf{p}}_\mathsf{O}) + \mathbf{F}_\beta(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) \tag{1}$$

and the polar decomposition of the deformation gradient

$$\mathbf{F}_\beta = \mathbf{R}_\beta \mathbf{U}_\beta. \tag{2}$$

The series expansions

$$\mathbf{R}_\beta = \mathbf{I} + \mathbf{\Theta}_\beta + o(\beta), \tag{3}$$
$$\mathbf{U}_\beta = \mathbf{I} + \mathbf{E}_\beta + o(\beta), \tag{4}$$

are made up of the sum of the value at $\beta = 0$, a linear term in β and the rest $o(\beta)$ such that

$$\lim_{\beta\to 0} \frac{o(\beta)}{\beta}\,\mathbf{a} = \mathbf{o} \quad \forall \mathbf{a} \in \mathcal{V}. \tag{5}$$

Substituting these espressions into (2) we obtain

$$\mathbf{F}_\beta = (\mathbf{I} + \mathbf{\Theta}_\beta)(\mathbf{I} + \mathbf{E}_\beta) + o(\beta) = \mathbf{I} + \mathbf{\Theta}_\beta + \mathbf{E}_\beta + o(\beta), \tag{6}$$

It is worth noting that $\mathbf{\Theta}_\beta$, called *infinitesimal rotation*, is a skew symmetric tensor, since

$$\mathbf{R}_\beta^\mathsf{T}\mathbf{R}_\beta = \mathbf{I} \quad \Rightarrow \quad (\mathbf{I} + \mathbf{\Theta}_\beta)^\mathsf{T}(\mathbf{I} + \mathbf{\Theta}_\beta) + o(\beta) = \mathbf{I} \quad \Rightarrow \quad \mathbf{\Theta}_\beta^\mathsf{T} + \mathbf{\Theta}_\beta + o(\beta) = \mathbf{O}, \tag{7}$$

while $\mathbf{E}_\beta$, called *infinitesimal stretch*, is a symmetric tensor like $\mathbf{U}_\beta$.

The deformation (1) can also be described by the displacement field

$$\mathbf{u}_\beta(\bar{\mathsf{p}}_\mathsf{A}) = \mathbf{u}_\beta(\bar{\mathsf{p}}_\mathsf{O}) + (\mathbf{F}_\beta - \mathbf{I})(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}), \tag{8}$$

which, by (6), becomes

$$\mathbf{u}(\bar{\mathsf{p}}_\mathsf{A}) = \mathbf{u}(\bar{\mathsf{p}}_\mathsf{O}) + (\mathbf{\Theta}_\beta + \mathbf{E}_\beta)(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) + o(\beta). \tag{9}$$

2 Infinitesimal stretch

By (4) the stretch of a segment parallel to $\mathbf{a}$ can be written as

$$\frac{\|\mathbf{U}_\beta\,\mathbf{a}\|}{\|\mathbf{a}\|} = \frac{1}{\|\mathbf{a}\|}(\mathbf{U}_\beta\,\mathbf{a}\cdot\mathbf{U}_\beta\,\mathbf{a})^{1/2} = \frac{1}{\|\mathbf{a}\|}(\|\mathbf{a}\| + \mathbf{E}_\beta\,\mathbf{a}\cdot\mathbf{a}) + o(\beta) = 1 + \frac{\mathbf{E}_\beta\,\mathbf{a}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}} + o(\beta). \tag{10}$$

Dropping the subscript β and denoting the matrix of $\mathbf{E}$ in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$[\mathbf{E}] = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}, \tag{11}$$

we get from (10)

$$\frac{\|\mathbf{U}\,\mathbf{e}_1\|}{\|\mathbf{e}_1\|} = 1 + \mathbf{E}\,\mathbf{e}_1\cdot\mathbf{e}_1 + o(\beta) = 1 + \varepsilon_{11} + o(\beta). \tag{12}$$

Hence up to $o(\beta)$ ε_{11} is the elongation in the direction of $\mathbf{e}_1$, ε_{22} is the elongation in the direction of $\mathbf{e}_2$, ε_{33} is the elongation in the direction of $\mathbf{e}_3$. Further, for the couple of basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ we get

$$\begin{aligned}\mathbf{U}\,\mathbf{e}_1\cdot\mathbf{U}\mathbf{e}_2 = \mathbf{U}^2\mathbf{e}_1\cdot\mathbf{e}_2 &= \big(\mathbf{I}+\mathbf{E}+o(\beta)\big)^2\mathbf{e}_1\cdot\mathbf{e}_2 = \big(\mathbf{I}+2\mathbf{E}+o(\beta)\big)\mathbf{e}_1\cdot\mathbf{e}_2\\ &= \mathbf{e}_1\cdot\mathbf{e}_2 + 2\mathbf{E}\,\mathbf{e}_1\cdot\mathbf{e}_2 + o(\beta) = 2\varepsilon_{21} + o(\beta).\end{aligned} \tag{13}$$

By using (12), after computing

$$\|\mathbf{U}\,\mathbf{e}_1\|\|\mathbf{U}\,\mathbf{e}_2\| = (1+\varepsilon_{11})(1+\varepsilon_{22}) + o(\beta) = 1+\varepsilon_{11}+\varepsilon_{22}+o(\beta) \tag{14}$$

$$(\|\mathbf{U}\,\mathbf{e}_1\|\|\mathbf{U}\,\mathbf{e}_2\|)^{-1} = 1-\varepsilon_{11}-\varepsilon_{22}+o(\beta), \tag{15}$$

eventually we get for the angle between $\mathbf{U}\,\mathbf{e}_1$ and $\mathbf{U}\,\mathbf{e}_2$

$$\cos\big(\frac{\pi}{2}-\gamma_{21}\big) = \frac{\mathbf{U}\,\mathbf{e}_1\cdot\mathbf{U}\,\mathbf{e}_2}{\|\mathbf{U}\,\mathbf{e}_1\|\|\mathbf{U}\,\mathbf{e}_2\|} = 2\varepsilon_{21}+o(\beta). \tag{16}$$

Since $\cos(\frac{\pi}{2}-\gamma_{21}) = \sin(\gamma_{21}) \simeq \gamma_{21}$, the *shear strain* γ_{21} turns out to be approximated by

$$\gamma_{21} \simeq 2\varepsilon_{21}. \tag{17}$$

By the same reason

$$\gamma_{32} \simeq 2\varepsilon_{32}, \quad \gamma_{13} \simeq 2\varepsilon_{13}. \tag{18}$$

It is worth noting that if $\mathbf{u}_i$ is an eigenvector of $\mathbf{E}$ and ε_i is the corresponding eigenvalue, we get

$$\mathbf{E}\mathbf{u}_i = \varepsilon_i\mathbf{u}_i \tag{19}$$

and by (4)

$$\mathbf{E}\mathbf{u}_i = (\mathbf{U}-\mathbf{I}+o(\beta))\mathbf{u}_i = \varepsilon_i\mathbf{u}_i \quad\Rightarrow\quad \mathbf{U}\mathbf{u}_i = (1+\varepsilon_i)\mathbf{u}_i + o(\beta). \tag{20}$$

Hence for a sufficiently small β the eigenvectors of $\mathbf{U}$ are close to the eigenvectors of $\mathbf{E}$, while the principal stretches are approximated by

$$\lambda_i \simeq 1+\varepsilon_i. \tag{21}$$

3 Infinitesimal rotations

The series expansion for the rotation can be conveniently derived in the following way. Let us consider a rotation as a composition of three elementary rotations (see Appendix 3)

$$\mathbf{R}_\beta = \mathbf{R}_\beta^{(3)}\mathbf{R}_\beta^{(2)}\mathbf{R}_\beta^{(1)} \tag{22}$$

whose axes are, respectively, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and whose amplitudes $\theta_\beta^{(1)}$, $\theta_\beta^{(2)}$, $\theta_\beta^{(3)}$, are linear functions of β, zero at $\beta=0$. Let us consider first $\mathbf{R}_\beta^{(1)}$. Its series expansion is

$$\mathbf{R}_\beta^{(1)} = \mathbf{I} + \mathbf{\Theta}_\beta^{(1)} + o(\beta) \tag{23}$$

corresponding to its matrix series expansion

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_\beta^{(1)} & -\sin\theta_\beta^{(1)}\\ 0 & \sin\theta_\beta^{(1)} & \cos\theta_\beta^{(1)}\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&-\theta_\beta^{(1)}\\0&\theta_\beta^{(1)}&0\end{pmatrix} + o(\beta). \tag{24}$$

Similar expansions can be derived for the other elementary rotations. By composing them we get

$$\mathbf{R}_\beta = (\mathbf{I}+\mathbf{\Theta}^{(3)}_\beta)(\mathbf{I}+\mathbf{\Theta}^{(2)}_\beta)(\mathbf{I}+\mathbf{\Theta}^{(1)}_\beta) + o(\beta) = \mathbf{I}+\mathbf{\Theta}^{(3)}_\beta+\mathbf{\Theta}^{(2)}_\beta+\mathbf{\Theta}^{(1)}_\beta + o(\beta). \tag{25}$$

Hence

$$\mathbf{\Theta}_\beta = \mathbf{\Theta}^{(3)}_\beta+\mathbf{\Theta}^{(2)}_\beta+\mathbf{\Theta}^{(1)}_\beta\,. \tag{26}$$

The matrices of $\mathbf{\Theta}^{(3)}_\beta$, $\mathbf{\Theta}^{(2)}_\beta$, $\mathbf{\Theta}^{(1)}_\beta$ turn out to be

$$\begin{pmatrix} 0 & -\theta^{(3)}_\beta & 0 \\ \theta^{(3)}_\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \theta^{(2)}_\beta \\ 0 & 0 & 0 \\ -\theta^{(2)}_\beta & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta^{(1)}_\beta \\ 0 & \theta^{(1)}_\beta & 0 \end{pmatrix}. \tag{27}$$

4 Volume change

By substitution of (4) in the expression for the volume of the parallelepiped with edges $\{\mathbf{U}_\beta\mathbf{e}_1, \mathbf{U}_\beta\mathbf{e}_2, \mathbf{U}_\beta\mathbf{e}_3\}$ we get

$$\begin{aligned} &\mathrm{vol}\,(\mathbf{U}_\beta\mathbf{e}_1, \mathbf{U}_\beta\mathbf{e}_2, \mathbf{U}_\beta\mathbf{e}_3) \\ &\quad = \mathrm{vol}\,((\mathbf{I}+\mathbf{E}_\beta)\mathbf{e}_1, (\mathbf{I}+\mathbf{E}_\beta)\mathbf{e}_2, (\mathbf{I}+\mathbf{E}_\beta)\mathbf{e}_3) + o(\beta) \\ &\quad = \mathrm{vol}\,(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) + \mathrm{vol}\,(\mathbf{E}_\beta\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) + \mathrm{vol}\,(\mathbf{e}_1,\mathbf{E}_\beta\mathbf{e}_2,\mathbf{e}_3) + \mathrm{vol}\,(\mathbf{e}_1,\mathbf{e}_2,\mathbf{E}_\beta\mathbf{e}_3) + o(\beta)\,, \end{aligned} \tag{28}$$

thus obtaining

$$\det \mathbf{F}_\beta = \frac{\mathrm{vol}\,(\mathbf{U}_\beta\mathbf{e}_1, \mathbf{U}_\beta\mathbf{e}_2, \mathbf{U}_\beta\mathbf{e}_3)}{\mathrm{vol}\,(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)} = 1 + \mathrm{tr}\,\mathbf{E}_\beta + o(\beta). \tag{29}$$

Hence, for β sufficiently small we find

$$\det \mathbf{F}_\beta \simeq 1 + \mathrm{tr}\,\mathbf{E}_\beta. \tag{30}$$

5 Area change

Let us consider a face $\mathcal{F}$ of a parallelepiped. The ratio between the area of that face and the area the face $\bar{\mathcal{F}}$ in the reference shape is given by

$$\frac{A_{\mathcal{F}}}{A_{\bar{\mathcal{F}}}} = \|(\mathrm{cof}\,\mathbf{F})\,\bar{\mathbf{n}}\| \tag{31}$$

where $\mathbf{n}$ is the exterior unit normal to $\bar{\mathcal{F}}$. From the series expansion of the expression above, for β sufficiently small we find

$$\|(\mathrm{cof}\,\mathbf{F}_\beta)\,\bar{\mathbf{n}}\| \simeq 1 + \mathrm{tr}\,\mathbf{E}_\beta - \mathbf{E}_\beta\bar{\mathbf{n}}\cdot\bar{\mathbf{n}}\,. \tag{32}$$

6 Linearized material response

The response function for an elastic material is

$$\mathbf{T}_\beta = \widehat{\mathbf{T}}(\mathbf{F}_\beta) = \mathbf{R}_\beta\,\widehat{\mathbf{T}}(\mathbf{U}_\beta)\,\mathbf{R}^{\mathsf{T}}_\beta. \tag{33}$$

If we assume

$$\widehat{\mathbf{T}}(\mathbf{I}) = \mathbf{O}, \tag{34}$$

we get the following series expansion

$$\widehat{\mathbf{T}}(\mathbf{U}_\beta) = \widehat{\mathbf{T}}(\mathbf{I} + \mathbf{E}_\beta) = \mathbb{C}(\mathbf{E}_\beta) + o(\beta), \tag{35}$$

where $\mathbb{C}$ transforms linearly a symmetric tensor into a symmetric tensor. Note that (33) becomes

$$\mathbf{T}_\beta = \widehat{\mathbf{T}}(\mathbf{F}_\beta) = (\mathbf{I} + \boldsymbol{\Theta}_\beta)^{\mathsf{T}}\mathbb{C}(\mathbf{E}_\beta)(\mathbf{I} + \boldsymbol{\Theta}_\beta) + o(\beta) = \mathbb{C}(\mathbf{E}_\beta) + o(\beta)\,. \tag{36}$$

7 Linearized total force and moment tensor

Along a trajectory described by a control parameter β, as in (1), the power of a force $\mathbf{f}_{\beta\mathsf{A}}$ applied at A is

$$\mathbf{f}_{\beta\mathsf{A}} \cdot \mathbf{v}_\mathsf{A} = \mathbf{f}_{\beta\mathsf{A}} \cdot \mathbf{v}_\mathsf{O} + \mathbf{F}_\beta(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) \otimes \mathbf{f}_{\beta\mathsf{A}} \cdot \mathbf{L}. \tag{37}$$

By assuming that the force $\mathbf{f}_{\beta\mathsf{A}}$ is linear in β and zero at $\beta = 0$, the series expansion of the moment tensor turns out to be

$$\mathbf{F}_\beta(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) \otimes \mathbf{f}_{\beta\mathsf{A}} = (\mathbf{I} + \mathbf{E}_\beta + \boldsymbol{\Theta}_\beta + o(\beta))(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) \otimes \mathbf{f}_{\beta\mathsf{A}} = (\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) \otimes \mathbf{f}_{\beta\mathsf{A}} + o(\beta)\,. \tag{38}$$

The power of a force distribution $\mathbf{b}_\beta$ on $\mathcal{R}_\beta$ in a velocity field $\mathbf{v}$ is the integral

$$\int_{\mathcal{R}_\beta} \mathbf{b}_\beta \cdot \mathbf{v}\, dV \tag{39}$$

which can be transformed into an integral on the shape $\bar{\mathcal{R}}$ by using the ratio between the volumes (change of variable formula)

$$\int_{\mathcal{R}_\beta} \mathbf{b}_\beta \cdot \mathbf{v}\, dV = \int_{\bar{\mathcal{R}}} (\mathbf{b}_\beta \circ \boldsymbol{\phi}) \cdot (\mathbf{v} \circ \boldsymbol{\phi}) \det \mathbf{F}_\beta\, dV \tag{40}$$

or, shortly,

$$\int_{\mathcal{R}} \mathbf{b}_\beta \cdot \mathbf{v}\, dV = \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{v} \det \mathbf{F}_\beta\, dV. \tag{41}$$

Replacing $\det \mathbf{F}_\beta$ with its series expansion (29) we get

$$\begin{aligned}\int_{\mathcal{R}_\beta} \mathbf{b}_\beta \cdot \mathbf{v}\, dV &= \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{v}\, (1 + \operatorname{tr} \mathbf{E}_\beta + o(\beta))\, dV \\ &= \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{v}\, dV + \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{v}\, \operatorname{tr} \mathbf{E}_\beta\, dV + o(\beta).\end{aligned} \tag{42}$$

Assuming that $\mathbf{b}_\beta$ is a linear function of β and is zero at $\beta = 0$ it turns out that

$$\int_{\mathcal{R}_\beta} \mathbf{b}_\beta \cdot \mathbf{v}\, dV = \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{v}\, dV + o(\beta) \tag{43}$$

because of the linearity of $\mathbf{E}_\beta$ in β. Further, from the expression for the affine test velocity field

$$\mathbf{v}(\mathsf{p}_\mathsf{A}) = \mathbf{v}_\mathsf{O} + \mathbf{L}\mathbf{F}_\beta(\bar{\mathsf{p}}_\mathsf{A} - \bar{\mathsf{p}}_\mathsf{O}) \tag{44}$$

we get, by using again (29),

$$\begin{aligned}\int_{\mathcal{R}_\beta} \mathbf{b}_\beta \cdot \mathbf{v}\, dV &= \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{v}_\mathsf{O}\, dV + \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta \cdot \mathbf{L}\mathbf{F}_\beta(\boldsymbol{x} - \bar{\mathsf{p}}_\mathsf{O})\, dV + o(\beta) \\ &= \int_{\bar{\mathcal{R}}} \mathbf{b}_\beta\, dV \cdot \mathbf{v}_\mathsf{O} + \int_{\bar{\mathcal{R}}} (\boldsymbol{x} - \bar{\mathsf{p}}_\mathsf{O}) \otimes \mathbf{b}_\beta\, dV \cdot \mathbf{L}\; + o(\beta).\end{aligned} \tag{45}$$

Let us assume also that $\mathbf{t}_\beta$, like $\mathbf{b}_\beta$, is a linear function of β and is zero at $\beta = 0$. Then, by using (32), we get

$$\begin{aligned}\int_{\partial\mathcal{R}_\beta} \mathbf{t}_\beta \cdot \mathbf{v}\, dA &= \int_{\partial\bar{\mathcal{R}}} \mathbf{t}_\beta \cdot \mathbf{v}\, \|(\operatorname{cof}\mathbf{F})\,\bar{\mathbf{n}}\|\, dA = \int_{\partial\bar{\mathcal{R}}} \mathbf{t}_\beta \cdot \mathbf{v}\, dA \\ &= \int_{\partial\bar{\mathcal{R}}} \mathbf{t}_\beta \cdot \mathbf{v}_{\mathsf{O}}\, dA + \int_{\partial\bar{\mathcal{R}}} \mathbf{t}_\beta \cdot \mathbf{L}\mathbf{F}_\beta(\boldsymbol{x} - \bar{\mathsf{p}}_{\mathsf{O}})\, dA + o(\beta) \\ &= \int_{\partial\bar{\mathcal{R}}} \mathbf{t}_\beta\, dA \cdot \mathbf{v}_{\mathsf{O}} + \int_{\partial\bar{\mathcal{R}}} (\boldsymbol{x} - \bar{\mathsf{p}}_{\mathsf{O}}) \otimes \mathbf{t}_\beta\, dA \cdot \mathbf{L} \; + o(\beta).\end{aligned} \tag{46}$$

8 Linear elasticity

The linear part of $\widehat{\mathbf{T}}(\mathbf{F}_\beta)$ given by (36) defines the response function in *linear elasticity*

$$\mathbf{T} = \mathbb{C}(\mathbf{E}). \tag{47}$$

The linear transformation $\mathbb{C}$ is called *elasticity tensor*. Since it transforms symmetric tensors into symmetric tensors, it will be described by a 6 by 6 matrix in any basis. In order for the strain energy to exist it can be proved that $\mathbb{C}$ has to be a symmetric tensor. Hence the total number of coefficients (elastic moduli) necessary to define the material response is $(6 \times 6 - 6)/2 + 6 = 21$. For *isotropic* materials that number reduces to 2 and the general form of the response function is

$$\mathbb{C}(\mathbf{E}) = \lambda \operatorname{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}. \tag{48}$$

The coefficients λ and μ are called Lamè's moduli. The infinitesimal rotation and the infinitesimal stretch are defined as

$$\boldsymbol{\Theta} := \operatorname{skw}(\mathbf{F} - \mathbf{I}) = \operatorname{skw} \nabla\mathbf{u}, \tag{49}$$

$$\mathbf{E} := \operatorname{sym}(\mathbf{F} - \mathbf{I}) = \operatorname{sym} \nabla\mathbf{u}, \tag{50}$$

where $\nabla\mathbf{u} = (\mathbf{F} - \mathbf{I})$ is the displacement gradient. An infinitesimal affine deformation is described by the expression

$$\phi(\bar{\mathsf{p}}_{\mathsf{A}}) = \phi(\bar{\mathsf{p}}_{\mathsf{O}}) + \mathbf{F}(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}) = \phi(\bar{\mathsf{p}}_{\mathsf{O}}) + (\mathbf{I} + \boldsymbol{\Theta} + \mathbf{E})(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}) \tag{51}$$

or, as an alternative, through the displacement gradient

$$\mathbf{u}(\bar{\mathsf{p}}_{\mathsf{A}}) = \mathbf{u}(\bar{\mathsf{p}}_{\mathsf{O}}) + \nabla\mathbf{u}(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}) = \mathbf{u}(\bar{\mathsf{p}}_{\mathsf{O}}) + (\boldsymbol{\Theta} + \mathbf{E})(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}})\,. \tag{52}$$

Let us summarize the linear elasticity theory for an affine body. The balance principle has the form

$$\mathbf{f} \cdot \mathbf{v}_{\mathsf{O}} + \left(\mathbf{M} - \mathbf{T}\, V_{\bar{\mathcal{R}}}\right) \cdot \mathbf{L} = 0 \quad \forall \mathbf{v}_{\mathsf{O}}, \forall \mathbf{L} \tag{53}$$

from which the following balance equations are derived

$$\mathbf{f} = \mathbf{o}, \tag{54}$$

$$\operatorname{skw} \mathbf{M} = \mathbf{O}, \tag{55}$$

$$\operatorname{sym} \mathbf{M} = \mathbf{T}\, V_{\bar{\mathcal{R}}}. \tag{56}$$

The total force and moment tensor are given by the expressions

$$\mathbf{f} = \int_{\bar{\mathcal{R}}} \mathbf{b}\, dV + \int_{\partial\bar{\mathcal{R}}} \mathbf{t}\, dA, \tag{57}$$

$$\mathbf{M}_{\mathsf{p}_{\mathsf{O}}} = \int_{\bar{\mathcal{R}}} (\boldsymbol{x} - \bar{\mathsf{p}}_{\mathsf{O}}) \otimes \mathbf{b}\, dV + \int_{\partial\bar{\mathcal{R}}} (\boldsymbol{x} - \bar{\mathsf{p}}_{\mathsf{O}}) \otimes \mathbf{t}\, dA \tag{58}$$

while the response function for the stress $\mathbf{T}$ is given by (47). Looking at the matrix of $\mathbf{E}$

$$[\mathbf{E}] = \begin{pmatrix} \varepsilon_{11} & \frac{\gamma_{12}}{2} & \frac{\gamma_{13}}{2} \\ \frac{\gamma_{21}}{2} & \varepsilon_{22} & \frac{\gamma_{23}}{2} \\ \frac{\gamma_{31}}{2} & \frac{\gamma_{32}}{2} & \varepsilon_{33} \end{pmatrix}, \tag{59}$$

the element ε_{11} stands for the elongation in the direction $\mathbf{e}_1$; the element γ_{12} stands for the shear corresponding to directions $\mathbf{e}_1$ and $\mathbf{e}_2$. Looking at the matrix of $\boldsymbol{\Theta}$

$$[\boldsymbol{\Theta}] = \begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}, \tag{60}$$

the three scalars θ_1, θ_2, θ_3 stand for the amplitudes of the three infinitesimal rotations around $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively.