Cauchy Continuum

Contents

1 Balance principle

Cauchy continuum is a body model whose placements are such that any deformation is a smooth map of the Euclidean space $\mathcal{E}$, without any of the restrictions like those characterizing affine or rigid bodies. Let us consider a body $\mathcal{B}$ and any placement

$$\mathsf{p} : \mathcal{B} \to \mathcal{E}. \tag{1}$$

In order to describe a *mechanical interaction* between the body and the exterior world we define a linear function $\mathcal{W}^{(out)}$, called *outer power* which, for any placement p, transforms any velocity field into a scalar.

The linear space of the *test velocity fields* is the collection of all the smooth velocity fields on $\mathcal{R}$. If in a test velocity field we denote the velocities at p_A and p_B by

$$\mathbf{v}_\mathsf{A}, \mathbf{v}_\mathsf{B} \tag{2}$$

the power has the representation

$$\mathcal{W}^{(out)}(\mathbf{v}_\mathsf{A}, \mathbf{v}_\mathsf{B}) = \mathbf{f}_\mathsf{A} \cdot \mathbf{v}_\mathsf{A} + \mathbf{f}_\mathsf{B} \cdot \mathbf{v}_\mathsf{B}. \tag{3}$$

The vectors $\mathbf{f}_\mathsf{A}$ and $\mathbf{f}_\mathsf{B}$ are called *external forces* applied to body points A and B respectively.

The case we are interested most is where $\mathcal{B}$ and p are such that the set $\operatorname{im}\mathsf{p}$ (the body *shape*) is a subset $\mathcal{R} \subset \mathcal{E}$ that is the closure of an open set, whose boundary $\partial\mathcal{R}$ is piecewise smooth. A test velocity field is a function

$$\mathbf{v} : \mathsf{p}_\mathsf{A} \mapsto \mathbf{v}_\mathsf{A}, \tag{4}$$

with domain $\mathcal{R}$. The *outer power* has in general the following representation

$$\mathcal{W}^{(out)}(\mathbf{v}) = \int_{\mathcal{R}} \mathbf{b} \cdot \mathbf{v}\, dV + \int_{\partial\mathcal{R}} \mathbf{t} \cdot \mathbf{v}\, dA. \tag{5}$$

The vector fields $\mathbf{b}$ and $\mathbf{t}$, on $\mathcal{R}$ and $\partial\mathcal{R}$ respectively, are called *bulk force distribution* and *surface force distribution (or tractions)*. The *inner power* is assumed in general to have the following representation

$$\mathcal{W}^{(in)}(\mathbf{v}) = -\int_{\mathcal{R}} (\mathbf{z} \cdot \mathbf{v} + \mathbf{T} \cdot \nabla\mathbf{v})\, dV, \tag{6}$$

where $\mathbf{z}$ and $\mathbf{T}$ are descriptors of the *stress* ($\mathbf{T}$ is called *Cauchy stress*) and $\nabla\mathbf{v}$ is the gradient of test velocity field.

We assume, as the *balance principle*, that *in any shape the* total *power is zero for any test velocity field*:

$$\mathcal{W}^{(out)}(\mathbf{v}) + \mathcal{W}^{(in)}(\mathbf{v}) = 0 \quad \forall\mathbf{v}. \tag{7}$$

2 Balance equations

By substituting the expression for the divergence of a tensor field (see Appendix 2)

$$\mathbf{T} \cdot \nabla\mathbf{v} = \operatorname{div}\left(\mathbf{T}^\mathsf{T}\mathbf{v}\right) - \operatorname{div}\mathbf{T} \cdot \mathbf{v} \tag{8}$$

in the expression for the inner power (6) we get

$$-\int_{\mathcal{R}} (\mathbf{z} \cdot \mathbf{v} + \mathbf{T} \cdot \nabla\mathbf{v})\, dV = -\int_{\mathcal{R}} \mathbf{z} \cdot \mathbf{v}\, dV + \int_{\mathcal{R}} \operatorname{div}\mathbf{T} \cdot \mathbf{v}\, dV - \int_{\mathcal{R}} \operatorname{div}\left(\mathbf{T}^\mathsf{T}\mathbf{v}\right) dV \tag{9}$$

which, by the divergence theorem, becomes

$$-\int_{\mathcal{R}} (\mathbf{z}\cdot\mathbf{v}+\mathbf{T}\cdot\nabla\mathbf{v})\,dV = -\int_{\mathcal{R}} \mathbf{z}\cdot\mathbf{v}\,dV + \int_{\mathcal{R}} \operatorname{div}\mathbf{T}\cdot\mathbf{v}\,dV - \int_{\partial\mathcal{R}} \big(\mathbf{T}^{\mathsf{T}}\mathbf{v}\big)\cdot\mathbf{n}\,dA$$
$$= -\int_{\mathcal{R}} \mathbf{z}\cdot\mathbf{v}\,dV + \int_{\mathcal{R}} \operatorname{div}\mathbf{T}\cdot\mathbf{v}\,dV - \int_{\partial\mathcal{R}} \mathbf{v}\cdot(\mathbf{T}\mathbf{n})\,dA \tag{10}$$

Hence the balance principle (7) can be given the form

$$\int_{\mathcal{R}} \big(\mathbf{b}-\mathbf{z}+\operatorname{div}\mathbf{T}\big)\cdot\mathbf{v}\,dV + \int_{\partial\mathcal{R}} \big(\mathbf{t}-\mathbf{T}\mathbf{n}\big)\cdot\mathbf{v}\,dA = 0 \quad \forall\mathbf{v}. \tag{11}$$

This is nothing but the weak form of the *balance equations*

$$\mathbf{b}-\mathbf{z}+\operatorname{div}\mathbf{T} = 0 \quad \text{in } \mathcal{R}, \tag{12}$$
$$\mathbf{t}-\mathbf{T}\mathbf{n} = 0 \quad \text{su } \partial\mathcal{R}. \tag{13}$$

3 Material objectivity

Differently than in the affine body model, the stress is a tensor field. We characterize the value of this tensor field at any position $\mathsf{p}_{\mathsf{A}} \in \mathcal{R}$, through the objectivity condition of the inner power density

$$\mathbf{z}(\mathsf{p}_{\mathsf{A}})\cdot\mathbf{v}(\mathsf{p}_{\mathsf{A}}) + \mathbf{T}(\mathsf{p}_{\mathsf{A}})\cdot\nabla\mathbf{v}(\mathsf{p}_{\mathsf{A}})\,. \tag{14}$$

Since this power depends only on the values at p_{A} of the velocity field and of its gradient, we can consider just affine velocity fields. The *material objectivity principle* reads: *the inner power density, for any affine test velocity field, is invariant under any change of framing*:

$$\mathbf{z}^*\cdot\mathbf{v}^*_{\mathsf{O}} + \mathbf{T}^*\cdot\mathbf{L}^* = \mathbf{z}\cdot\mathbf{v}_{\mathsf{O}} + \mathbf{T}\cdot\mathbf{L} \tag{15}$$

where we dropped the position, as an argument, where the stress is evaluated.

Since this statement is the same as for an affine body, the same are the conclusions

$$\mathbf{z} = \mathbf{Q}_c^{\mathsf{T}}\mathbf{z}^* = \mathbf{o}, \tag{16}$$
$$\mathbf{T} = \mathbf{Q}_c^{\mathsf{T}}\mathbf{T}^*\mathbf{Q}_c, \tag{17}$$
$$\operatorname{skw}\mathbf{T} = \mathbf{O}\,. \tag{18}$$

As a consequence the *balance equations* becomes

$$\mathbf{b}+\operatorname{div}\mathbf{T} = 0 \quad \text{in } \mathcal{R}, \tag{19}$$
$$\mathbf{t}-\mathbf{T}\mathbf{n} = 0 \quad \text{su } \partial\mathcal{R}. \tag{20}$$

4 Material response

By the *localization principle* the response function has the same form and the same characterization as for the affine bodies. The only difference is that the response function can be non uniform. When it is uniform the material is said to be *homogeneous*.