$(37-38)_7$ Monday [2014-04-07]
16:00 - 18:00 A13

Material response

$$T = \hat{T}(F) \quad \text{elastic material}$$

response function

$$p_\Delta = p_o + F(\bar{p}_\Delta - \bar{p}_o)$$

$$p_\Delta^* = q_o^* + Q(p_\Delta - p_o) \quad \text{change of observer}$$

$$p_o^* = q_o^* + Q(p_o - p_o) = q_o^*$$

$$p_\Delta^* - p_o^* = Q(p_\Delta - p_o) = QF(\bar{p}_\Delta - \bar{p}_o) = F^*(\bar{p}_\Delta - \bar{p}_o)$$

$$T^* = Q T Q^T \qquad T^* = \hat{T}(F^*) \qquad F^* = QF$$

$$\hat{T}(F^*) = Q\hat{T}(F)Q^T \quad \forall Q$$

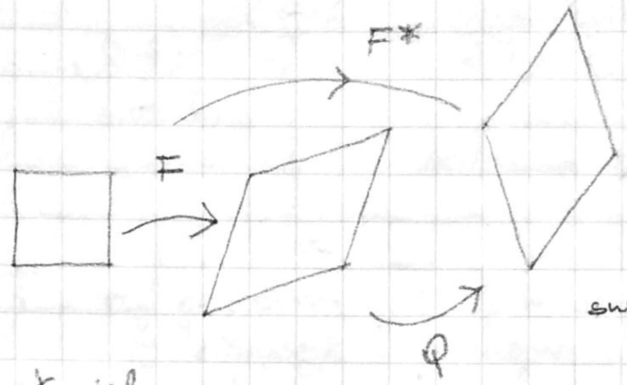
$$\hat{T}(QF) = Q\hat{T}(F)Q^T \quad \forall Q, \forall F$$

$$\hat{T}(QRU) = Q\hat{T}(RU)Q^T \quad \forall Q$$

$$Q = R^T \Rightarrow \quad \hat{T}(U) = R^T\hat{T}(F)R$$

$$\hat{T}(F) = R\hat{T}(U)R^T$$

objective response function reduced form

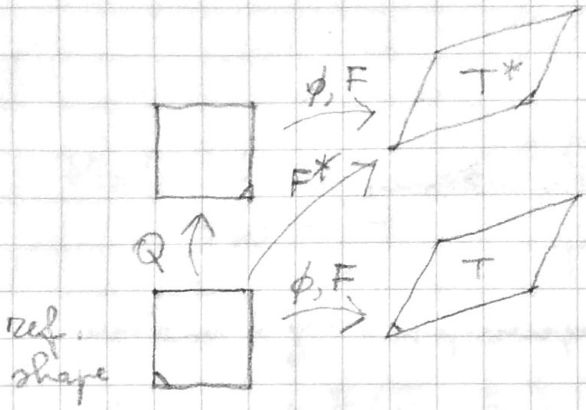


$$F^* = QF$$

superimposed rigid motion

material objectivity

$$\hat{T}(QF) = Q\hat{T}(F)Q^T$$



$$F^* = FQ \Rightarrow T^* = \hat{T}(FQ)$$

$$T = \hat{T}(F)$$

material symmetry

$$\hat{T}(F) = \hat{T}(FQ) \qquad \hat{T} \text{ is } Q \text{ invariant}$$

Symmetry group: collection of such rotations Q

Isotropic material: symmetry group $Orth^+$

Isotropy

$$\hat{T}(F) = \hat{T}(FQ) = \hat{T}(RUQ) = \hat{T}(\overbrace{R\underbrace{QQ^T}_{I}}^{\tilde{R}}\overbrace{UQ}^{\tilde{U}}) \qquad \forall Q$$

objectivity

$$= (RQ)\hat{T}(Q^TUQ)(RQ)^T$$

$$= R\,Q\hat{T}(Q^TUQ)Q^TR^T$$

$$\hat{T}(F) = \hat{T}(RU) = R\hat{T}(U)R^T$$

$$\Rightarrow \quad Q\hat{T}(Q^TUQ)Q^T = \hat{T}(U)$$

$$\Leftrightarrow \quad \hat{T}(Q^TUQ) = Q^T\hat{T}(U)Q \qquad \forall Q$$

Isotropic response function

$(39\text{-}40)_7$ Tuesday [2014-04-08]
9:00-11:00

$$T \cdot L \, V_{\mathcal{R}} = T \cdot L \, V_{\bar{\mathcal{R}}} \det F$$

$$L(p_A - p_o) = LF(\bar{p}_A - \bar{p}_o)$$

$$T \cdot L \, V_{\bar{\mathcal{R}}} \det F = S \cdot LF \, V_{\bar{\mathcal{R}}}$$

$$= SF^T \cdot L \, V_{\bar{\mathcal{R}}}$$

$$T \det F = SF^T$$

$$S = T F^{-T} \det F$$

PIOLA stress

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\bar{\mathcal{R}}} (T \cdot \nabla v) \det F \, dV = \int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, dV$$

$$\nabla \bar{v} = \nabla v \, F \quad \Rightarrow \quad T \cdot \nabla v \det F = T \cdot \nabla \bar{v} F^{-1} \det F$$

$$= (\det F)(T F^{-T}) \cdot \nabla \bar{v} = S \cdot \nabla \bar{v}$$

$$\operatorname{div}(S^T \bar{v}) = S \cdot \nabla \bar{v} + \operatorname{div} S \cdot \bar{v}$$

fields on $\bar{\mathcal{R}}$

$$\int_{\mathcal{R}} b \cdot v \, dV = \int_{\bar{\mathcal{R}}} (b \cdot v) \det F \, dV = \int_{\bar{\mathcal{R}}} \bar{b} \cdot \bar{v} \, dV$$

$$\bar{v}(x) = v(\varphi(x)) \, , \quad \bar{b} = b \det F$$

$$\int_{\partial\mathcal{R}} t \cdot v \, dA = \int_{\partial\bar{\mathcal{R}}} \bar{t} \cdot \bar{v} \, d\Delta \qquad \bar{t} = t \, \frac{\Delta_{\mathcal{F}}}{\Delta_{\bar{\mathcal{F}}}}$$

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, d\Delta$$

$$\int_{\bar{\mathcal{R}}} \bar{b} \cdot \bar{v} \, dV + \int_{\partial\bar{\mathcal{R}}} \bar{t} \cdot \bar{v} \, dV - \int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, d\Delta = 0 \qquad \forall \bar{v}$$

[...]
$$\int_{\bar{\mathcal{R}}} \operatorname{div}(S^T \bar{v}) \, dV = \int_{\partial\bar{\mathcal{R}}} S^T \bar{v} \cdot \bar{n} \, dV$$

outward unit normal

[...]
$$\begin{cases} \bar{b} + \operatorname{div} S = 0 & \bar{\mathcal{R}} \\ \bar{t} - S \bar{n} = 0 & \partial\bar{\mathcal{R}} \end{cases}$$

compare
$$t = Tn \, , \quad \bar{t} = S \bar{n} \qquad S = T \operatorname{cof} F$$

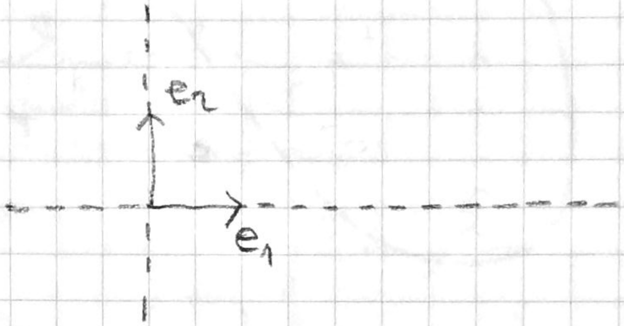
$(41-42)_7$ Wednesday [2014-04-09]

9:00 – 11:00 A1,2

Scalar form of the balance equations

$$\operatorname{div} T \cdot e_i = \operatorname{div}(T^T e_i) = \operatorname{tr} \nabla (T^T e_i)$$

$$[T] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

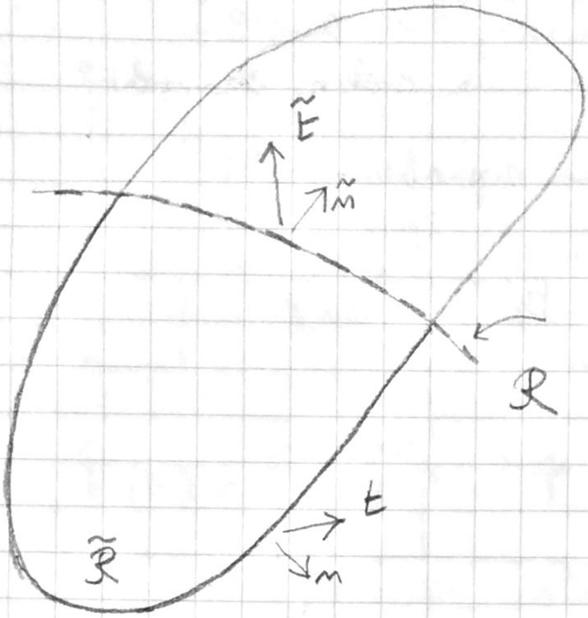


$$T^T e_1 = \sigma_{11}\, e_1 + \sigma_{12}\, e_2 + \sigma_{13}\, e_3$$
$$T^T e_2 = \sigma_{21}\, e_1 + \sigma_{22}\, e_2 + \sigma_{23}\, e_3$$
$$T^T e_3 = \sigma_{31}\, e_1 + \sigma_{32}\, e_2 + \sigma_{33}\, e_3$$

$$[\nabla(T^T e_1)] = \begin{pmatrix} \sigma_{11,1} & \sigma_{11,2} & \sigma_{11,3} \\ \sigma_{12,1} & \sigma_{12,2} & \sigma_{12,3} \\ \sigma_{13,1} & \sigma_{13,2} & \sigma_{13,3} \end{pmatrix}$$

$$\operatorname{div} T \cdot e_1 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}$$

$$\operatorname{div} T \cdot e_i = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3}$$



$$\operatorname{div} T + b = 0$$

$$T n = t$$

Cauchy cut

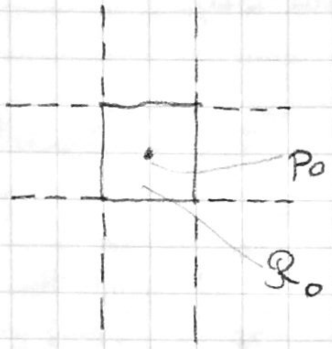
$\mathcal{R}$ an ideal splitting

$$\int_{\tilde{\mathcal{R}}} b \cdot v \, dV + \int_{\partial\tilde{\mathcal{R}}} t \cdot v \, dA - \int_{\tilde{\mathcal{R}}} T \cdot \nabla v \, dV = 0$$

[...] $\operatorname{div} T + b = 0$

$$T n = t \,, \quad T\tilde{n} = \tilde{t}$$

$\tilde{t}$ is the traction we should apply to keep the lower part of the body balanced after removing the upper part.



we can cut a sample in the shape of a cube centered at p_o : $\mathcal{R}_o \subset \mathcal{R}$

By using an affine test velocity field we get

$$\int_{\mathcal{R}_o} T \cdot \nabla v \, dV = \int_{\mathcal{R}_o} T \cdot L \, dV = \left(\int_{\mathcal{R}_o} T \, dV \right) \cdot L$$

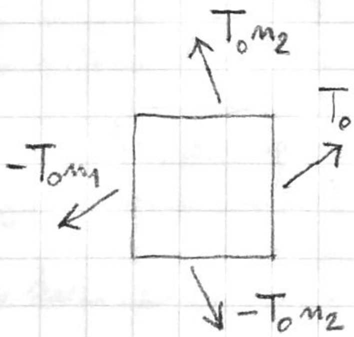
If we compare this expression with the one we wrote for an affine body we find

stress in the affine body

$$T_o \cdot L \, V_{\mathcal{R}_o} = \int_{\mathcal{R}_o} T \, dV \cdot L$$

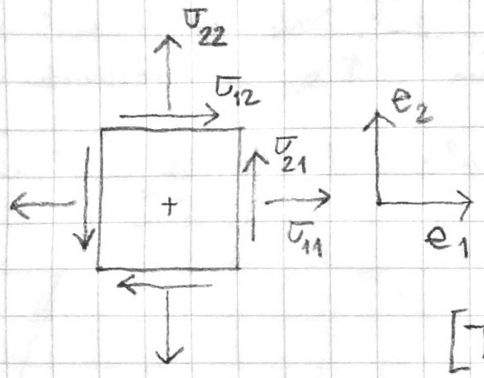
stress field

$$\Rightarrow \quad T_o = \frac{1}{V_{\mathcal{R}_o}} \int_{\mathcal{R}_o} T \, dV \qquad \text{mean stress}$$



traction on the boundary of the cube balanced by the uniform stress

$$T_o \simeq T(p_o) \quad (\text{small } V_{\mathcal{R}_o})$$



$$
\begin{array}{cccc}
 & Te_1 & Te_2 & Te_3 \\
[T]_e = & \multicolumn{3}{c}{\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}}
\end{array}
$$

$T = T^T$

$\Rightarrow \quad \sigma_{21} = \sigma_{12} \quad [\cdots]$

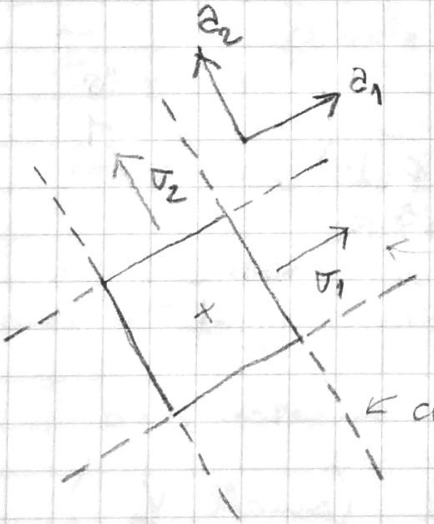
$\Downarrow$

real eigenvalues and orthogonal eigenvectors

$\sigma_1, \sigma_2, \sigma_3$ $\qquad\qquad$ a_1, a_2, a_3

principal stresses $\qquad\qquad$ principal directions

diagonal matrix $\quad [T]_a = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}$



normal stress
no tangential stress

cuts orthogonal to the principal directions