$(75-76)_{14}$ Monday [2014-05-26] A1.3
16:00 - 18:00

Linearization around the reference shape

Let us consider a one-parameter family of deformations

$$F(\beta) = R(\beta)U(\beta) \qquad \beta \text{ dimensionless}$$

$$R(\beta) = I + \beta \frac{d}{d\beta} R(\beta)\Big|_{\beta=0} + o(\beta)$$

$$U(\beta) = I + \beta \frac{d}{d\beta} U(\beta)\Big|_{\beta=0} + o(\beta)$$

$$R(\beta)^T R(\beta) = I \quad \Rightarrow \quad \frac{d}{d\beta}\left(R(\beta)^T R(\beta)\right) = 0$$

$$\Rightarrow \quad \frac{d}{d\beta} R(\beta)^T\Big|_{\beta=0} + \frac{d}{d\beta} R(\beta)\Big|_{\beta=0} = 0$$

infinitesimal rotation

$$\Theta := \frac{d}{d\beta} R(\beta)\Big|_{\beta=0} \quad \Rightarrow \quad \Theta^T + \Theta = 0$$

infinitesimal stretch

$$E := \frac{d}{d\beta} U(\beta)\Big|_{\beta=0} \quad \Rightarrow \quad E^T = E$$

$$F(\beta) = I + \beta\Theta + \beta E + o(\beta)$$

 

$(77\text{-}78)_{14}$ Tuesday [2014-05-27] A1.3
9:00 – 11:00

Following the formula about time differentiation already derived $\to (17\text{-}18)_3$

we get

$$\det F(\beta) = \det F(0) + \beta \frac{d}{d\beta} \det F(\beta)\Big|_{\beta=0} + o(\beta)$$

$$\frac{d}{d\beta} \det F(\beta)\Big|_{\beta=0} = \det F(\beta)\, \mathrm{tr}\Big(\big(\frac{d}{d\beta} F(\beta)\big) F(\beta)^{-1}\Big)\Big|_{\beta=0}$$

$$= \mathrm{tr}\Big((\Theta + E)(I - \beta\Theta - \beta E)\Big)\Big|_{\beta=0}$$

$$= \mathrm{tr}\,(\Theta + E) = \mathrm{tr}\, E$$

$$\det F(\beta) = 1 + \beta\, \mathrm{tr}\, E + o(\beta)$$

In view of linearization it is customary to define the displacement vector field

$$u(\bar{p}_\Delta) := \phi(\bar{p}_\Delta) - \bar{p}_\Delta \quad \forall\, \bar{p}_\Delta \in \bar{\mathcal{R}}$$

Along any curve $\bar{c}$ we get

$$u(\bar{c}(h)) = \phi(\bar{c}(h)) - \bar{c}(h)$$
$$u(\bar{c}(0)) = \phi(\bar{c}(0)) - \bar{c}(0)$$

 

$$\frac{u(\bar{c}(h)) - u(\bar{c}(0))}{h} = \frac{\phi(\bar{c}(h)) - \phi(\bar{c}(0))}{h} - \frac{\bar{c}(h) - \bar{c}(0)}{h}$$

Taking the limit as $h \to 0$ we arrive at

$$\nabla u \, c' = \nabla \phi \, c' - c'$$

Hence

$$\nabla u = F - I$$

By using the corresponding series expansions we get

$$\nabla u(\beta) = F(\beta) - I = \beta \left( \Theta + E \right) + o(\beta)$$

$$\lim_{\beta \to 0} \frac{\nabla u(\beta)}{\beta} = \Theta + E$$

Since there is a unique decomposition of any Tensor into the sum of a skewsymmetric and a symmetric Tensor, by setting

$$\Theta := \operatorname{skw} \nabla u \,, \quad E := \operatorname{sym} \nabla u$$

we get

$$\nabla u(\beta) = \beta \left( \operatorname{skw} \nabla u + \operatorname{sym} \nabla u \right) + o(\beta) = \beta \nabla u + o(\beta)$$

In the linear theory we use quantities like $\Theta, E, \nabla u$ which are derivatives with respect to $\beta$, a dimensionless "scaling factor" for some control parameters like forces.

 

For the response function of an elastic material we write the series expansion

$$\hat{T}(F(\beta)) = \hat{T}(I) + \beta \frac{d}{d\beta}\hat{T}(F(\beta))\Big|_{\beta=0} + o(\beta)$$

where, because of the objectivity condition

$$\hat{T}(F) = R\,\hat{T}(U)\,R^T \qquad \rightarrow (37\text{-}38)_7$$

we have

$$\frac{d}{d\beta}\hat{T}(F(\beta))\Big|_{\beta=0} = \frac{d}{d\beta} R(\beta)\,\hat{T}(U(\beta))\,R(\beta)^T\Big|_{\beta=0}$$

$$= \underset{\substack{\downarrow\\ 0}}{W\hat{T}(I)} + \frac{d}{d\beta}\hat{T}(U(\beta)) - \underset{\substack{\downarrow\\ 0}}{\hat{T}(I)W}$$

with

$$\frac{d}{d\beta}\hat{T}(U(\beta)) = \frac{d}{d\beta}\hat{T}(I+\beta E) = \mathbb{C}E$$

where $\mathbb{C} : Sym(\mathcal{V}) \rightarrow Sym(\mathcal{V})$ is the gradient of of a tensor field over the space of symmetric tensors. It is referred to as the elasticity tensor.

Notice how

$$\frac{d}{d\beta}\hat{T}(U(\beta))\Big|_{\beta=0} = \lim_{\beta\to 0}\frac{1}{\beta}\left(\hat{T}(U(\beta)) - \hat{T}(U(0))\right) = \mathbb{C}E$$

with

$$E = \frac{d}{d\beta}U(\beta)\Big|_{\beta=0}$$

looks like

$$(19\text{-}20)_4 \leftarrow \qquad \lim_{h\to 0}\frac{1}{h}\left(v(c(h)) - v(c(0))\right) = \nabla v\, c'$$

$(79\text{-}80)_{14}$ Wednesday [2014-05-28] A1.3
9:00 – 11:00

Let us recall the definition of the Piola stress

$$S = (\det F)\, T\, F^{-T} \qquad \to (39\text{-}40)_7$$

Then the first order term in the series expansion of

$$S(\beta) = (\det F(\beta))\, T(\beta)\, F(\beta)^{-T}$$

will be

$$\frac{d}{d\beta} S(\beta)\Big|_{\beta=0} = (\operatorname{tr} E)\, T(0)\, F(0)^{-T} + (\det F(0)) \frac{d}{d\beta} T(\beta)\, F(0)^{-T}$$
$$+ (\det F(0))\, T(0)\, (\Theta - E)$$

$$T(0) = 0 \quad \Rightarrow \quad \frac{d}{d\beta} S(\beta)\Big|_{\beta=0} = \frac{d}{d\beta} T(\beta)$$

$$\Rightarrow \quad S(\beta) = T(\beta) + o(\beta)$$

By this property, for a hyperelastic material the relation $\quad \hat{S}(F(\beta)) \cdot \dot{F}(\beta) = \frac{d}{dt} \varphi(F(\beta))$

can be rewritten as

$$o(\beta) + \hat{T}(F(\beta)) \cdot \dot{F}(\beta) = \frac{d}{dt} \varphi(F(\beta))$$

and then, by the results on the previous pages,

$$o(\beta^2) + (\beta\, \mathbb{C}(E)) \cdot (\beta \dot{\Theta} + \beta \dot{E}) = \frac{d}{dt} \varphi(U(\beta))$$

where $\varphi(F(\beta)) = \varphi(U(\beta))$ because of objectivity.

 

By the symmetry of $\mathbb{C}(E)$ we get

$$o(\beta^2) + \beta^2\, \mathbb{C}(E)\cdot\dot{E} = \frac{d}{dt}\varphi(U(\beta))$$

For a potential to exist the stress power on the left side should be such that

$$\mathbb{C}(E)\cdot\dot{E} = \mathbb{C}(\dot{E})\cdot E$$

since $\mathbb{C} : Sym(\mathcal{V}) \to Sym(\mathcal{V})$ is a linear function

That means that the elasticity tensor should be symmetric

$$\mathbb{C}E\cdot\dot{E} = E\cdot\mathbb{C}^T\dot{E} = E\cdot\mathbb{C}\dot{E}$$

Hence the strain energy is, up to second order, given by

$$\varphi(E) = \frac{1}{2}\,\mathbb{C}(E)\cdot E$$

Since $\dim(Sym(\mathcal{V})) = 6$, the elasticity tensor matrix is a $6\times 6$ matrix. As a symmetric matrix (in an orthonormal tensor basis) it has only 21 independent entries $\left(\frac{1}{2}(6\times 6-6)+6\right)$, describing material properties.

It is amazing to find out that by linearizing the general response function for isotropic hyperelastic materials $\to (45\text{-}46)_8$ we get $\mathbb{C}(E) = \lambda_L (\mathrm{tr}\,E) I + 2\mu_L E$

where, by isotropy, there are only two coefficients left, $\lambda_L$ and $\mu_L$, called the Lamé constants (or moduli).

The symmetry property of the elasticity tensor for a hyperelastic material can be derived through the following detailed computation.

Let

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ \dots \\ \dots \\ \dots \\ \dots \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{pmatrix}$$

be the component form of the linear elasticity response

$$T = \mathbb{C} E$$

where both $T$ and $E$ are symmetric tensors.

The stress power can be written by using the above components as

$$T \cdot \dot{E} = \mathbb{C}(E) \cdot \dot{E}$$

$$= c_{11}\, \varepsilon_{11}\, \dot{\varepsilon}_{11} + c_{12}\, \varepsilon_{12}\, \dot{\varepsilon}_{11} + c_{13}\, \varepsilon_{13}\, \dot{\varepsilon}_{11} + \dots$$

$$+ c_{21}\, \varepsilon_{11}\, \dot{\varepsilon}_{12} + c_{22}\, \varepsilon_{12}\, \dot{\varepsilon}_{12} + c_{23}\, \varepsilon_{13}\, \dot{\varepsilon}_{12} + \dots$$

$$+ c_{31}\, \varepsilon_{11}\, \dot{\varepsilon}_{13} + c_{32}\, \varepsilon_{12}\, \dot{\varepsilon}_{13} + c_{33}\, \varepsilon_{13}\, \dot{\varepsilon}_{13} + \dots$$

$$+ \dots$$

from which it is clear how $c_{12} = c_{21}$, $c_{13} = c_{31}$, $c_{23} = c_{32}$ etc are the conditions for a primitive to exist.

The linearized response function for an isotropic hyperelastic material can be derived from the general expression → $(45\text{-}46)_8$ by differentiating $\hat{T}(F(\beta))$ with respect to $\beta$.

Since

$$\frac{d}{d\beta} F(\beta)\Big|_{\beta=0} = W + E$$

$$\frac{d}{d\beta} B(\beta)\Big|_{\beta=0} = \frac{d}{d\beta} F(\beta) + \frac{d}{d\beta} F(\beta)^T = 2E$$

$$\frac{d}{d\beta} B^2(\beta)\Big|_{\beta=0} = 4E \quad , \quad \frac{d}{d\beta} B(\beta)^{-1}\Big|_{\beta=0} = -2E$$

$$\frac{d}{d\beta} \iota_3(\beta)\Big|_{\beta=0} = \det B(\beta)\, \operatorname{tr}\left(\frac{d}{d\beta} B(\beta)\; B(\beta)^{-1}\right)\Big|_{\beta=0} = 2 \operatorname{tr} E$$

$$\frac{d}{d\beta} \iota_1(\beta)\Big|_{\beta=0} = 2 \operatorname{tr} E \quad , \quad \frac{d}{d\beta} \iota_2(\beta)\Big|_{\beta=0} = 4 \operatorname{tr} E$$

we get an expression as a linear combination of the two tensors $(\operatorname{tr} E) I$ and $E$

which we write

$$\mathbb{C}(E) = \lambda_L\, (\operatorname{tr} E) I + 2\mu_L\, E$$