

## LITHIUM ION BATTERIES

Molar concentration of Lithium as an interstitial species (of an anode particle)

$$c = \frac{\rho_{Li}}{\rho_0} = \frac{\text{molar density of Lithium atoms per unit ref. volume}}{\text{molar density of lattice sites per unit ref. volume}}$$

Host material lattice deformation

$$\phi : \mathcal{R}_0 \rightarrow \mathcal{R} \quad F_0 := \nabla \phi$$

The total amount of host material (moles) is

$$\int_{\mathcal{R}_0} \rho_0 dV = \int_{\mathcal{R}_0} \rho \det F_0 dV = \int_{\mathcal{R}} \rho dV \quad \text{with} \quad \frac{d}{dt} \rho_0 = 0$$

$$\text{and} \quad \int_{\mathcal{R}_0} \rho_{Li} dV = \int_{\mathcal{R}_0} c \rho_0 dV = \int_{\mathcal{R}} c \frac{\rho_0}{\det F_0} dV = \int_{\mathcal{R}} c \rho dV$$

is the total amount of Lithium (moles). Its rate of change in time turns out to be

$$\frac{d}{dt} \int_{\mathcal{R}} c \rho dV = \frac{d}{dt} \int_{\mathcal{R}_0} c \rho_0 dV = \int_{\mathcal{R}_0} \dot{c} \rho_0 dV = \int_{\mathcal{R}} \dot{c} \rho dV$$

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Species molar balance

$$\frac{d}{dt} \int_{\mathcal{R}} c \rho dV = - \int_{\partial \mathcal{R}} h \cdot n dA + \int_{\mathcal{R}} h dV$$

$h$  flux                   $h$  source (reaction)

$$\int_{\mathcal{R}} \dot{c} \rho dV = - \int_{\mathcal{R}} \operatorname{div} h \quad (\text{no source})$$

$$\dot{c} \rho = - \operatorname{div} h$$

Species power balance

$$\int_{\mathcal{R}} \mu \dot{c} \rho dV = - \int_{\mathcal{R}} \mu \operatorname{div} h dV$$

$\mu$  energy per mole of Lithium

$$\begin{aligned} \operatorname{div}(\mu h) &= \operatorname{tr} \nabla(\mu h) = \operatorname{tr}(\mu \nabla h + \nabla \mu \otimes h) \\ &= \mu \operatorname{div} h + \nabla \mu \cdot h \end{aligned}$$

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$$\int_{\mathcal{R}} \mu \dot{\epsilon} \rho \, dV = - \int_{\mathcal{R}} \operatorname{div}(\mu h) \, dV + \int_{\mathcal{R}} h \cdot \nabla \mu \, dV$$

$$\int_{\mathcal{R}} \mu \dot{\epsilon} \rho \, dV = - \int_{\partial \mathcal{R}} \mu h \cdot n \, dA + \int_{\mathcal{R}} h \cdot \nabla \mu \, dV$$

Pull back to the reference shape

$$\int_{\partial \mathcal{R}} \mu h \cdot n \, dA = \int_{\partial \mathcal{R}_0} \mu h \cdot \operatorname{cof} F_0 \, n_0 \, dA$$

$$= \int_{\partial \mathcal{R}_0} \mu h \cdot F_0^{-T} n_0 (\det F_0) \, dA$$

$$= \int_{\partial \mathcal{R}_0} \mu F_0^{-1} h \cdot n_0 (\det F_0) \, dA$$

$$= \int_{\partial \mathcal{R}_0} \mu_0 h_0 \cdot n_0 \, dA$$

$$h_0 := (\det F_0) F_0^{-1} h$$

$$\mu_0(x) = \mu(\phi(x))$$

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$$\int_{\mathcal{R}} h \cdot \nabla \mu \, dV = \int_{\mathcal{R}} h \cdot F_0^{-T} \nabla \mu_0 \, dV$$

$$\nabla \mu \cdot F_0 e = \nabla \mu_0 \cdot e$$

$$F_0^T \nabla \mu = \nabla \mu_0 \quad \Rightarrow \quad \nabla \mu = F_0^{-T} \nabla \mu_0$$

$$\int_{\mathcal{R}} h \cdot \nabla \mu \, dV = \int_{\mathcal{R}_0} F_0^{-1} h \cdot \nabla \mu_0 \det F_0 \, dV = \int_{\mathcal{R}_0} h_0 \cdot \nabla \mu_0 \, dV$$

Species power balance in  $\mathcal{R}_0$ 

$$\int_{\mathcal{R}_0} \mu_0 \dot{c} \rho_0 \, dV = - \underbrace{\int_{\partial \mathcal{R}_0} \mu_0 h_0 \cdot n_0 \, d\Delta}_{\text{external}} + \int_{\mathcal{R}_0} h_0 \cdot \nabla \mu_0 \, dV$$

$\mu_0(x) = \mu(\phi(x))$

Force power balance in  $\mathcal{R}_0$ 

$$\underbrace{\int_{\mathcal{R}_0} b_0 \cdot n_0 \, dV + \int_{\partial \mathcal{R}_0} t_0 \cdot n_0 \, d\Delta}_{\text{external}} = \int_{\mathcal{R}_0} S_0 \cdot \nabla n_0 \, dV$$

$n_0(x) = n(\phi(x))$

Free energy imbalance (localized)

$$S_0 \cdot \dot{F}_0 + \mu_0 \dot{c} \rho_0 - h_0 \cdot \nabla \mu_0 - \frac{d}{dt} \psi \geq 0$$

Let us assume for the free energy density per unit reference volume

$$\psi = \hat{\psi}(F_0, c)$$

and

$$\dot{\psi} = S_0 \cdot \dot{F}_0 + \mu_0 \dot{c} \rho_0$$

The free energy imbalance reduces to

$$-h_0 \cdot \nabla \mu_0 \geq 0$$

If  $h_0 = -M \nabla \mu_0$  Fick's law

with  $M u \cdot u \geq 0$  ( $M$  positive semi-definite tensor)

then  $-h_0 \cdot \nabla \mu_0 = M \nabla \mu_0 \cdot \nabla \mu_0 \geq 0$

$M$  is called the mobility tensor. If  $M = dI$  ( $d \geq 0$ )

$d$  is called the diffusion coefficient.

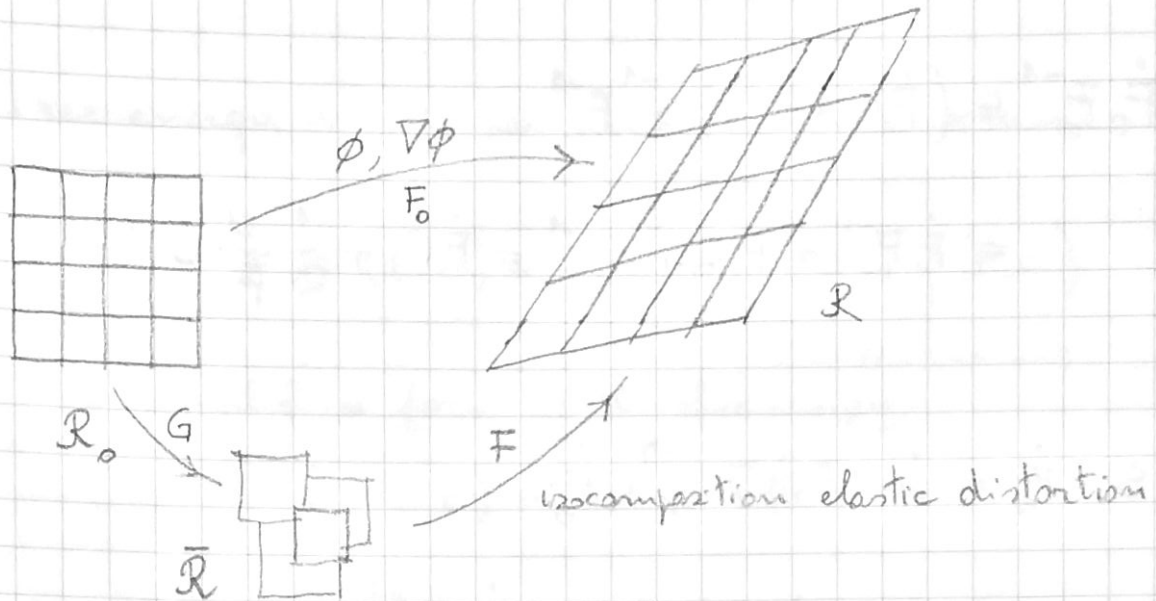
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Note

Avogadro constant:  $6.02 \dots 10^{23}$

It's the number of atoms in 12 gr of carbon  $^{12}\text{C}$ .

A mole is a quantity of matter made up of an Avogadro number of atoms or molecules.



$$\phi: \mathcal{R}_0 \rightarrow \mathcal{R} \quad \nabla\phi = FG \quad F_0 \equiv \nabla\phi$$

$$\det G = \beta(c) \quad G = \beta(c)^{\frac{1}{3}} I \quad (*)$$

$$\dot{G} = \frac{1}{3} \beta(c)^{-\frac{2}{3}} \frac{d\beta}{dc} \dot{c} \quad \dot{G} = \frac{1}{3} \beta^{-\frac{2}{3}} \alpha \dot{c} I$$

$$\dot{G} G^{-1} = \frac{1}{3} \beta^{-\frac{2}{3}} \beta^{-\frac{1}{3}} \alpha \dot{c} I = \frac{1}{3} \frac{\alpha}{\beta} \dot{c} I$$

$$(\det G) \dot{G} G^{-1} = \frac{1}{3} \alpha \dot{c} I \quad \alpha = \frac{d\beta}{dc}$$

$$\dot{F}_0 = \dot{F} G + F \dot{G} = \beta^{\frac{1}{3}} \left( \dot{F} + \frac{1}{3} \frac{\alpha}{\beta} \dot{c} F \right)$$

(\*) Denoting by  $V_{Li}$  the molar volume increment due to Li intercalation:

$$\det G = (\rho_0 V_0 + \rho_{Li} V_{Li}) / (\rho_0 V_0) = 1 + c \frac{V_{Li}}{V_0} = 1 + c\alpha = \beta(c)$$

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$$\begin{aligned}\dot{F}_0 F_0^{-1} &= (\dot{F}G + F\dot{G})G^{-1}F^{-1} \\ &= \dot{F}F^{-1} + F\dot{G}G^{-1}F^{-1} = \dot{F}F^{-1} + \frac{1}{3} \frac{\alpha}{\beta} \dot{c} I\end{aligned}$$

$$\begin{aligned}S_0 \cdot \dot{F}_0 &= (\det F_0) T F_0^{-T} \cdot \dot{F}_0 \\ &= (\det F)(\det G) T \cdot \dot{F}_0 F_0^{-1} \\ &= \beta (\det F) T \cdot \dot{F}F^{-1} + \frac{1}{3} \alpha (\det F) T \cdot I \dot{c} \\ &= \beta S \cdot \dot{F} + \frac{1}{3} \alpha (\det F)(\text{tr} T) \dot{c}\end{aligned}$$

Thus the free energy imbalance becomes

$$\begin{aligned}\beta S \cdot \dot{F} + \frac{1}{3} \alpha (\det F)(\text{tr} T) \dot{c} + \mu_0 \dot{\rho}_0 \\ - h_0 \cdot \nabla \mu_0 - \frac{d}{dt} \psi \geq 0\end{aligned}$$

Note that this expression depends on the special decomposition of the deformation gradient defined in the diagram above.



Free energy density per unit reference volume

$$\psi = \hat{\psi}(F_0, c) = \rho_0 \varphi_0(c) + \det G \varphi(F, c)$$

The simplified form of the free energy

$$\hat{\psi}(F_0, c) = \det G \varphi(F)$$

together with the assumption

$$\frac{d}{dt} \varphi(F) = S \cdot \dot{F} \quad \Rightarrow \quad S = \hat{S}(F)$$

$$S_0 = \hat{S}(F) G^{-T} \det G = \hat{S}(F) \beta^{\frac{2}{3}}$$

leads to

$$\frac{d}{dt} \psi = \frac{d}{dt} (\det G) \varphi(F) + (\det G) \frac{d}{dt} \varphi(F)$$

$$= (\det G) \operatorname{tr}(\dot{G} G^{-1}) \varphi(F) + (\det G) S \cdot \dot{F}$$

$$= \alpha \dot{c} \varphi(F) + \beta S \cdot \dot{F}$$

Hence the free energy imbalance turns into

$$\left( -\alpha \varphi(F) + \frac{1}{3} \alpha (\det F) (\operatorname{tr} T) + \mu_0 \rho_0 \right) \dot{c} - h_0 \cdot \nabla \mu_0 \geq 0$$

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In order for the inequality to hold for any evolution (constitutive process) it must be

$$-\alpha \varphi(F) + \frac{1}{3} \alpha (\det F) (\operatorname{tr} T) + \mu_0 \rho_0 = 0$$

$$-h_0 \cdot \nabla \mu_0 \geq 0$$

From the first condition we get

$$\mu_0 = \frac{\alpha}{\rho_0} \left( \varphi(F) - \frac{1}{3} (\det F) (\operatorname{tr} T) \right) \quad (*)$$

which is a constitutive characterization of  $\mu_0$  (the chemical potential)

From the second condition we get

$$h_0 = -M \nabla \mu_0 \quad (\text{Fick's law})$$

with  $M$  a positive semi-definite tensor.

(\*) Notice that the expression between brackets is just

$$\frac{1}{3} \operatorname{tr} (\varphi(F) I - (\det F) T)$$

defining the spherical part of the Eshelby tensor.

## POWER BALANCE SUMMARY

Force power balance

$$\int_{\mathcal{R}_0} b_0 \cdot v_0 \, dV + \int_{\partial \mathcal{R}_0} t_0 \cdot v_0 \, dA - \int_{\mathcal{R}_0} S_0 \cdot \nabla v_0 \, dV = 0 \quad \forall v_0$$

Species power balance

$$\int_{\mathcal{R}_0} \mu_0 c \, \rho_0 \, dV + \int_{\partial \mathcal{R}_0} \mu_0 h_0 \cdot n_0 \, dA - \int_{\mathcal{R}_0} h_0 \cdot \nabla \mu_0 \, dV = 0 \quad \forall \mu_0$$

## CONSTITUTIVE CHARACTERIZATION

$$S_0 = \beta^{\frac{2}{3}} \hat{S}(F)$$

$$\beta = \hat{\beta}(c) \quad (\beta = 1 + \alpha c)$$

$$h_0 = \hat{h}_0(\mu_0) = -M \nabla \mu_0 \quad \tilde{h}_0$$

$$\mu_0 = \hat{\mu}_0(F) = \frac{\alpha}{\rho_0} \left( \psi(F) - \frac{1}{3} (\det F) (\text{tr} T) \right) \quad \tilde{\mu}_0$$

Unknown fields

$$u, c, (S_0, \beta, h_0, \mu_0)$$

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There are two possible ways of operating a battery:

- a. Potentiostatic operation: the electrode is surrounded by a constant lithium concentration;
- b. Galvanostatic operation: the current, thus the ionic flux at the surface of the electrode, is a constant.

[Deshpande - Cheng - Verbruggen, 2010]

#### BALANCE EQUATION SUMMARY

$$\operatorname{div} S_o + b_o = 0 \quad \text{in } R_o$$

$$S_o n_o = t_o \quad \text{on } \partial R_o$$

$$\operatorname{div} h_o + \dot{c} p_o = 0 \quad \text{in } R_o$$

$$c = \bar{c} \quad \text{on } \partial R_o \quad (\text{potentiostatic})$$

$$h_o = \bar{h}_o \quad \text{on } \partial R_o \quad (\text{galvanostatic})$$

APPENDIX (nothing new; only variant derivations)

The host material lattice deformation  $\phi$  leaves the host material contents in  $\mathcal{R}_0$ , or any subset  $\mathcal{P}_0 \subset \mathcal{R}_0$ , unchanged:

denoting by  $\rho$  the density of lattice sites per unit current volume

$$\int_{\mathcal{P}} \rho \, dV = \int_{\mathcal{P}_0} \rho_0 \, dV \quad \mathcal{R}_0 \supset \mathcal{P}_0 \xrightarrow{\phi} \mathcal{P} \subset \mathcal{R}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \, dV = \frac{d}{dt} \int_{\mathcal{P}_0} \rho \det F_0 \, dV = \frac{d}{dt} \int_{\mathcal{P}_0} \rho_0 \, dV = 0 \quad \forall \mathcal{P}_0 \subset \mathcal{R}_0$$

$$\Rightarrow \frac{d}{dt} \rho_0 = 0 \quad \frac{d}{dt} (\rho \det F_0) = 0$$

$$\dot{\rho} \det F_0 + \rho \frac{d}{dt} \det F_0 = 0$$

$$(\dot{\rho} + \rho \operatorname{tr}(\dot{F}_0 F_0^{-1})) \det F_0 = 0$$

$$\dot{\rho} + \rho \operatorname{div} v = 0$$

molar  
conservation law  
for lattice sites

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The quantity of intercalated lithium will change in time

$$\frac{d}{dt} \int_{\mathcal{R}} c \rho \, dV = \frac{d}{dt} \int_{\mathcal{R}_0} c \rho \det F_0 \, dV$$

$$= \int_{\mathcal{R}_0} (\dot{c} \rho + c \dot{\rho} + c \rho \operatorname{div} v) \det F_0 \, dV$$

$$= \int_{\mathcal{R}} (\dot{c} \rho + \underbrace{c (\dot{\rho} + \rho \operatorname{div} v)}_0) \, dV = \int_{\mathcal{R}} \dot{c} \rho \, dV$$

$$= \int_{\mathcal{R}_0} \dot{c} \rho_0 \, dV$$

Species molar balance in  $\mathcal{R}$  and any  $\partial \mathcal{R}$

$$\int_{\mathcal{R}} \dot{c} \rho \, dV = - \int_{\partial \mathcal{R}} h \cdot n \, dA$$

$$\Rightarrow \dot{c} \rho = - \operatorname{div} h$$

Pull back to the reference shape  $\mathcal{R}_0$

$$\int_{\mathcal{R}} \dot{c} \rho \, dV = \int_{\mathcal{R}_0} \dot{c} \rho_0 \, dV$$

$$-\int_{\partial \mathcal{R}} h \cdot n \, dA = -\int_{\partial \mathcal{R}_0} h \cdot F_0^{-T} n_0 (\det F_0) \, dA = -\int_{\partial \mathcal{R}_0} h_0 \cdot n_0 \, dA$$

with

$$h_0 = (\det F_0) F_0^{-T} h$$

Species molar balance in  $\mathcal{R}_0$  and any  $\mathcal{P}_0 \subset \mathcal{R}_0$

$$\int_{\mathcal{R}_0} \dot{c} \rho_0 \, dV = -\int_{\partial \mathcal{R}_0} h_0 \cdot n_0 \, dA$$

$$\Rightarrow \dot{c} \rho_0 = -\operatorname{div} h_0$$

In order to turn the species molar balance into a power balance let us introduce a scalar field  $\mu$  in  $\mathcal{R}$  and the scalar field  $\mu_0$  in  $\mathcal{R}_0$  such that

$$\mu_0(x) = \mu(\phi(x))$$

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Then

$$\nabla \mu \quad \mu \dot{\rho} = -\mu \operatorname{div} h \quad \text{in } \mathcal{R}$$

$$\nabla \mu_0 \quad \mu_0 \dot{\rho}_0 = -\mu_0 \operatorname{div} h_0 \quad \text{in } \mathcal{R}_0$$

$$\int_{\mathcal{R}_0} \mu_0 \dot{\rho}_0 dV = - \int_{\mathcal{R}_0} \mu_0 \operatorname{div} h_0 dV$$

$$\int_{\mathcal{R}_0} \mu_0 \dot{\rho}_0 dV = - \int_{\mathcal{R}_0} \operatorname{div} (\mu_0 h_0) dV + \int_{\mathcal{R}_0} h_0 \cdot \nabla \mu_0 dV$$

$$\int_{\mathcal{R}_0} \mu_0 \dot{\rho}_0 dV = - \int_{\partial \mathcal{R}_0} \mu_0 h_0 \cdot n_0 dV + \int_{\mathcal{R}_0} h_0 \cdot \nabla \mu_0 dV$$

[→ p. 4]



General formula for

$$\operatorname{div}(\alpha v)$$

with  $\alpha$  a scalar field and  $v$  a vector field

$$\operatorname{div}(\alpha v) = \operatorname{tr}(\nabla(\alpha v))$$

$$\nabla(\alpha v) c' = \lim_{h \rightarrow 0} \frac{1}{h} \left( \alpha(c(h)) v(c(h)) - \alpha(c(0)) v(c(0)) \right) \\ + \alpha(c(h)) v(c(0)) - \alpha(c(h)) v(c(0))$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left( \alpha(c(h)) - \alpha(c(0)) \right) v(c(0))$$

$$+ \lim_{h \rightarrow 0} \frac{1}{h} \alpha(c(h)) \left( v(c(h)) - v(c(0)) \right)$$

$$= \underbrace{(\nabla \alpha c')} v + \alpha (\nabla v c')$$

scalar field which can be written as  $(\nabla \alpha \cdot c')$

$$= (\nabla \alpha \cdot c') v + (\alpha \nabla v) c'$$

$$= (v \otimes \nabla \alpha) c' + (\alpha \nabla v) c'$$

$$\Rightarrow \nabla(\alpha v) = \alpha \nabla v + v \otimes \nabla \alpha$$

$$\operatorname{div}(\alpha v) = \operatorname{tr} \nabla(\alpha v) = \alpha \operatorname{div} v + v \cdot \nabla \alpha$$

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Physical dimensions and units of measure

molar density  $[\rho_0] = \text{mol m}^{-3}$

from  $\mu_0 = \frac{\alpha}{\rho_0} \left( \varphi(F) - \frac{1}{3} (\det F) \text{tr} T \right)$

we get  $\frac{1}{\alpha} \rho_0 \mu_0$  as an energy density per unit relaxed volume

Since  $\alpha$  is a dimensionless quantity then

$$[\rho_0 \mu_0] = \underbrace{(\text{Pa m}^2 \text{m})}_{\text{energy}} \text{m}^{-3} = \text{Pa}$$

$$[\mu_0] = \text{Pa mol}^{-1} \text{m}^3 = \text{Pa m}^2 \text{m mol}^{-1}$$

Since the concentration  $c$  is a dimensionless quantity then

$$[c \rho_0] = \frac{1}{s} \text{mol m}^{-3}$$

From the molar balance we get

$$\int_{\mathcal{R}_0} \dot{c} \rho_0 dV = - \int_{\partial \mathcal{R}_0} h_0 \cdot n_0 dA \Rightarrow [h_0] = \frac{1}{s} \text{mol m}^{-2}$$

Since

$$[\nabla\mu_0] = \text{Pa} \cdot \text{m}^3 \cdot \text{mol}^{-1} \cdot \text{m}^{-1} = \text{Pa} \cdot \text{m}^2 \cdot \text{mol}^{-1}$$

from the Fick's law  $h_0 = -M \nabla\mu_0$

we get

$$[M] = \left(\frac{1}{\text{s}} \cdot \text{mol} \cdot \text{m}^{-2}\right) (\text{Pa}^{-1} \cdot \text{m}^2 \cdot \text{mol})$$

$$= \frac{1}{\text{s}} \cdot \text{mol}^2 \cdot \text{Pa}^{-1} \cdot \text{m}^{-4}$$

Square brackets have been used for denoting units of measure in the SI system.

As an alternative the energy can be measured by J (Joule) and Pa (Pascal) can be replaced by N (Newton) per  $\text{m}^2$ :

$$1 \text{ Pa} = 1 \text{ N} \cdot \text{m}^{-2}$$

$$1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ Pa} \cdot \text{m}^2 \cdot \text{m}$$

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## APPENDIX (Derivatives of the free energy)

[from p. 9]

$$\frac{d}{dt} \hat{\psi}(F_0, c) = \alpha \varphi(F) \dot{c} + \beta S \cdot \dot{F}$$

[from p. 8]

$$\beta S \cdot \dot{F} = -\frac{1}{3} \alpha (\det F) (\operatorname{tr} T) \dot{c} + S_0 \cdot \dot{F}_0$$

Hence

$$\frac{d}{dt} \hat{\psi}(F_0, c) = \alpha \left( \varphi(F) - \frac{1}{3} (\det F) (\operatorname{tr} T) \right) \dot{c} + S_0 \cdot \dot{F}_0$$

[from p. 10]

$$\frac{d}{dt} \hat{\psi}(F_0, c) = l_0 \mu_0 \dot{c} + S_0 \cdot \dot{F}_0$$

That is why we can say that, as a result, the chemical potential turns out to be the derivative of the free energy with respect to the concentration

$$\frac{d}{dc} \psi(F_0, c) = l_0 \mu_0$$

We can also write for short

$$\frac{\partial}{\partial F_0} \hat{\psi}(F_0, c) = S_0$$

APPENDIX (How  $\mu_0$  depends on  $c$ )

$$\mu_0 = \frac{\alpha}{\rho_0} \left( \varphi(F) - \frac{1}{3} (\det F) \operatorname{tr} T \right)$$

Because the basic descriptors are the deformation  $\chi$  and the intercalation distortion  $G$ , the elastic distortion  $F$  is related to them by

$$F = F_0 G^{-1} = F_0 \beta^{-1/3} \quad \beta = 1 + \alpha c$$

Further

$$S = T F^{-T} (\det F) \Rightarrow \hat{T}(F) = \hat{S}(F) F^T (\det F)^{-1}$$

$$\varphi(F) = k_I (\bar{I}_1 - 3) + k_V (\det F - 1)^2$$

$$\bar{I}_1 = I_1 (\det F)^{-2/3}$$

$$I_1 = \operatorname{tr}(F^T F) = \operatorname{tr}(F_0^T F_0) \beta^{-2/3}$$

$$\det F = (\det F_0) / \beta$$

Hence  $\mu_0$  depends on  $F$  and in turn on  $c$  through  $\beta$ .

Even if the material is incompressible ( $\det F = 1$ )

$$\text{then } \det F_0 = \det G = \beta$$

$$\varphi(F) = k_I (I_1 - 3) ; \quad -\frac{1}{3} \operatorname{tr} T = p$$

$$\mu_0 = \frac{\alpha}{\rho_0} \left( \varphi(F) + p \right)$$

## APPENDIX (Darcy's Law)

Let us consider an interstitial species whose molar volume in a natural state is  $V^m$ , and define the discharge per unit reference area

$$q_0 = h_0 V^m$$

Substituting the Fick's Law

$$h_0 = -d \nabla \mu_0$$

and the expression we got for  $\mu_0$ , we arrive at

$$q_0 = - \left( d \frac{\alpha}{\rho_0} V^m \right) \nabla \left( \varphi(F) - \frac{1}{3} (\det F) \operatorname{tr} T \right)$$

which can be seen as a Darcy's Law suitable for a fluid flowing through a porous isotropic elastic material.

Unit of measure for the discharge per unit area

$$\begin{aligned} [q_0] &= [h_0] \text{ m}^3 \text{ mol}^{-1} \\ &= \frac{1}{s} \text{ mol m}^{-2} \text{ m}^3 \text{ mol}^{-1} = \frac{\text{m}}{s} \end{aligned}$$

For the total discharge  $Q_0 = q_0 \underset{\text{area}}{A}$  we get  $[Q_0] = \frac{\text{m}^3}{s}$ .