(25-26) [2016-03-31]

Velocity gradient

- rigid velocity field

$$\nabla v = \dot{R} R^T$$

$$W \qquad W^T = -W$$

- affine velocity field

$$\nabla v = \dot{F} F^T$$

$$L = \underbrace{W}_{\text{skew symmetric part}} + \underbrace{D}_{\text{symmetric part}}$$

Force distribution

$\mathcal{F}(v)$ $\qquad$ v test velocity field

general representation

$$\mathcal{F}(v) = \int_{\mathcal{R}} b(x)\cdot v(x)\, dV + \int_{\partial\mathcal{R}} t(x)\cdot v(x)\, dA$$

Singular distribution

$$\mathcal{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + \dots$$

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Affine velocity test fields

$$v(x) = v_o + L\,(x - p_o)$$

$$\begin{aligned}\mathcal{F}(v) = {} & f_A \cdot v_o + f_B \cdot v_o + f_C \cdot v_o \\ & + f_A \cdot L\,(p_A - p_o) \\ & + f_B \cdot L\,(p_B - p_o) \\ & + f_C \cdot L\,(p_C - p_o)\end{aligned}$$

$$f \cdot L\,(x - p_o) = f \otimes (x - p_o) \cdot L$$

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(27-28) [2016-04-04]

$$\operatorname{tr} A := \frac{\operatorname{vol}(A e_1, e_2, e_3) + \operatorname{vol}(e_1, A e_2, e_3) + \ldots}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$A e_j = a_{ij} e_i$$

$$A e_1 = a_{i1} e_i$$

$$A e_2 = a_{i2} e_i$$

$$A e_3 = a_{i3} e_i$$

$$\operatorname{vol}(a_{i1} e_i, e_2, e_3) = a_{11} \operatorname{vol}(e_1, e_2, e_3)$$

$$\operatorname{vol}(e_1, a_{i2} e_i, e_3) = a_{22} \operatorname{vol}(\qquad)$$

$$\operatorname{vol}(e_1, e_2, a_{i3} e_i) = a_{33} \operatorname{vol}(\qquad)$$

$$\operatorname{tr} A = a_{11} + a_{22} + a_{33}$$

The definition above turns out to be independent of the basis $\{e_1, e_2, e_3\}$

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$$\mathrm{tr}\,(u\otimes v) = \frac{\mathrm{vol}((u\otimes v)e_1, e_2, e_3) + \mathrm{vol}(\quad) + \mathrm{vol}(\quad)}{\mathrm{vol}(e_1, e_2, e_3)}$$

$$\mathrm{vol}((u\otimes v)e_1, e_2, e_3) = (v\cdot e_1)\,\mathrm{vol}(u, e_2, e_3)$$

$$= (v\cdot e_1)\,u_1\,\mathrm{vol}(e_1, e_2, e_3)$$

$$= v\cdot(u_1 e_1)\,\mathrm{vol}(e_1, e_2, e_3)$$

$\boxed{\ldots}$

$$\mathrm{tr}(u\otimes v) = u\cdot v$$

$$\begin{cases} f\cdot Lu = \mathrm{tr}(f\otimes Lu) = \mathrm{tr}((f\otimes u)L^T) \\ f\cdot Lu = L^T f\cdot u = \mathrm{tr}(L^T f\otimes u) = \mathrm{tr}(L^T(f\otimes u)) \end{cases}$$

$$f\cdot Lu = Lu\cdot f = \mathrm{tr}(Lu\otimes f) = \mathrm{tr}(L(u\otimes f))$$

$$f\cdot Lu = u\cdot L^T f = \mathrm{tr}(u\otimes L^T f) = \mathrm{tr}((u\otimes f)L)$$

defining $\quad M\cdot L = \mathrm{tr}(ML^T)$

we get also $\quad M\cdot L = \mathrm{tr}(L^T M)$

$$M\cdot L = \mathrm{tr}(LM^T)$$

$$M\cdot L = \mathrm{tr}(M^T L)$$

These are equivalent definitions

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$\{e_1, e_2, e_3\}$ orthonormal basis

$$AB = AB(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)$$

$$\begin{aligned}
\operatorname{tr} AB &= \operatorname{tr}(ABe_1 \otimes e_1 + ABe_2 \otimes e_2 + ABe_3 \otimes e_3)\\
&= ABe_1 \cdot e_1 + ABe_2 \cdot e_2 + ABe_3 \cdot e_3\\
&= Be_1 \cdot A^T e_1 + Be_2 \cdot A^T e_2 + Be_3 \cdot A^T e_3\\
&= \operatorname{tr}(Be_1 \otimes A^T e_1) + \operatorname{tr}(Be_2 \otimes A^T e_2) + \operatorname{tr}(Be_3 \otimes A^T e_3)\\
&= \operatorname{tr}(Be_1 \otimes A^T e_1 + Be_2 \otimes A^T e_2 + Be_3 \otimes A^T e_3)\\
&= \operatorname{tr}(B(e_1 \otimes e_1)A + B(e_2 \otimes e_2)A + B(e_3 \otimes e_3)A)\\
&= \operatorname{tr}(B(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)A) = \operatorname{tr}(BA)
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr} A &= \operatorname{tr}(Ae_1 \otimes e_1 + Ae_2 \otimes e_2 + Ae_3 \otimes e_3)\\
&= Ae_1 \cdot e_1 + Ae_2 \cdot e_2 + Ae_3 \cdot e_3\\
&= e_1 \cdot A^T e_1 + e_2 \cdot A^T e_2 + e_3 \cdot A^T e_3\\
&= A^T e_1 \cdot e_1 + A^T e_2 \cdot e_2 + A^T e_3 \cdot e_3\\
&= \operatorname{tr}(A^T e_1 \otimes e_1 + A^T e_2 \otimes e_2 + A^T e_3 \otimes e_3) = \operatorname{tr} A^T
\end{aligned}$$

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(29-30) [2016-04-05]

$$\boxed{M \cdot L = \mathrm{tr}(M L^T)} \quad ①$$

$$\boxed{M \cdot L = \mathrm{tr}(M^T L)} \quad ②$$

$L = W + D$

$$M \cdot W \underset{①}{=} \mathrm{tr}(M W^T) = -\mathrm{tr}(M W) \underset{②}{=} -M^T \cdot W$$

$$\boxed{W^T = -W}$$

$$\Rightarrow \quad (M + M^T) \cdot W = 0$$

$$M \cdot D \underset{①}{=} \mathrm{tr}(M D^T) = \mathrm{tr}(M D) \underset{②}{=} M^T \cdot D$$

$$\boxed{D^T = D}$$

$$\Rightarrow \quad (M - M^T) \cdot D = 0$$

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[2016-04-04]

$$A e_j = a_{ij} e_i \qquad e_i \cdot e_j = \delta_{ij}$$

$$A e_k = a_{ik} e_i$$ orthonormal basis

$$a_{ij}(e_i \otimes e_j)$$

$$a_{ij}(e_i \otimes e_j) e_k = a_{ij}\, e_i\, \delta_{jk} = a_{ik}\, e_i = A e_k \quad \forall k$$

$$\Downarrow$$

$$A = a_{ij}(e_i \otimes e_j)$$

$$A \cdot B = \mathrm{tr}(AB^T)$$

$$B = b_{hk}(e_h \otimes e_k)$$

$$B^T = b_{hk}(e_h \otimes e_k)^T = b_{hk}(e_k \otimes e_h)$$

$$AB^T = a_{ij}\, b_{hk}(e_i \otimes e_j)(e_k \otimes e_h)$$

$$(e_i \otimes e_j)(e_k \otimes e_h) w = (e_i \otimes e_j)\, e_k (e_h \cdot w) = e_i\, \delta_{jk}(e_h \cdot w) = \delta_{jk}(e_i \otimes e_h) w$$

$$\mathrm{tr}(e_i \otimes e_h) = e_i \cdot e_h = \delta_{ih} \Rightarrow \mathrm{tr}(AB^T) = a_{ij}\, b_{ij}$$

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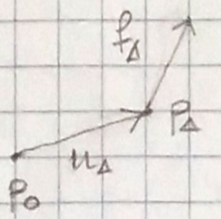
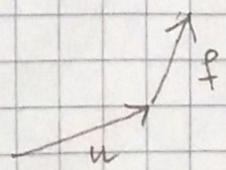
[2016-04-05]

$$f_\Delta \cdot v(p_\Delta) = f_\Delta \cdot v_o + \underbrace{f_\Delta \cdot L(p_\Delta - p_o)}$$

$$\big(f_\Delta \otimes \underbrace{(p_\Delta - p_o)}_{u_\Delta}\big) \cdot L$$

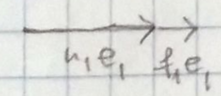


$$v(p_\Delta) = v_o + L(p_\Delta - p_o)$$

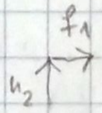


$$(f \otimes u)\, e_j = f\,(u \cdot e_j) \qquad \text{(orthonormal basis)}$$

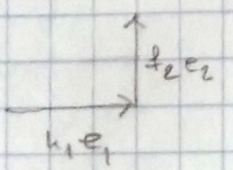
$$[f \otimes u] = \begin{pmatrix} f_1 u_1 & f_1 u_2 & f_1 u_3 \\ f_2 u_1 & f_2 u_2 & f_2 u_3 \\ f_3 u_1 & f_3 u_2 & f_3 u_3 \end{pmatrix}$$



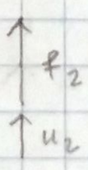
$f_1 u_1 (e_1 \otimes e_1)$



$f_1 u_2 (e_1 \otimes e_2)$



$f_2 u_1 (e_2 \otimes e_1)$



$f_2 u_2 (e_2 \otimes e_2)$

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$$
\begin{pmatrix}
f_1 u_1 & & \\
\frac{1}{2}(f_2 u_1 + f_1 u_2) & f_2 u_2 & \\
\frac{1}{2}(f_3 u_1 + f_1 u_3) & \frac{1}{2}(f_3 u_2 + f_2 u_3) & f_3 u_3
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & & \\
\frac{1}{2}(f_2 u_1 - f_1 u_2) & 0 & \\
\frac{1}{2}(f_3 u_1 - f_1 u_3) & \frac{1}{2}(f_3 u_2 - f_2 u_3) & 0
\end{pmatrix}
$$

$$
[W] = \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix}
$$

$$
[D] = \begin{pmatrix}
d_{11} & & \\
d_{21} & d_{22} & \\
d_{31} & d_{32} & d_{33}
\end{pmatrix}
$$

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[2016-04-05]

$$v_o + W(p_A - p_o) = v_o + W u_A$$

$$
\begin{aligned}
W u &= W(u_1 e_1 + u_2 e_2 + u_3 e_3) \\
&= u_1(\omega_3 e_2 - \omega_2 e_3) \\
&\quad + u_2(-\omega_3 e_1 + \omega_1 e_3) \\
&\quad + u_3(\omega_2 e_1 - \omega_1 e_2) \\
&= (-\omega_3 u_2 + \omega_2 u_3)\, e_1 \\
&\quad + (\omega_3 u_1 - u_3 \omega_1)\, e_2 \\
&\quad + (-u_1 \omega_2 + u_2 \omega_1)\, e_3
\end{aligned}
$$

$$
\begin{aligned}
f \cdot W u &= f_1(-\omega_3 u_2 + \omega_2 u_3) \\
&\quad + f_2(\omega_3 u_1 - u_3 \omega_1) \\
&\quad + f_3(-u_1 \omega_2 + u_2 \omega_1) \\
&= \omega_3(-f_1 u_2 + f_2 u_1) \\
&\quad + \omega_2(f_1 u_3 - f_3 u_1) \\
&\quad + \omega_1(-f_2 u_3 + f_3 u_2)
\end{aligned}
$$

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$$(f \otimes u) \cdot W = f_i u_j W_{ij}$$

with

$$W_{ij} = -\varepsilon_{ijk} \omega_k \qquad \text{(Levi-Civita symbol)}$$

$$\varepsilon_{ijk} = \begin{cases} 0 & !(i \neq j \neq k) \\ +1 & \forall \text{ even permutation of } 123 \\ -1 & \forall \text{ odd permutation of } 123 \end{cases}$$

$$W_{12} = -\varepsilon_{123} \omega_3 = -\omega_3 \,, \quad W_{21} = -\varepsilon_{213} \omega_3 = \omega_3$$

$$W_{13} = -\varepsilon_{132} \omega_2 = \omega_2 \,, \quad W_{31} = -\varepsilon_{312} \omega_2 = -\omega_2$$

$$W_{23} = -\varepsilon_{231} \omega_1 = -\omega_1 \,, \quad W_{32} = -\varepsilon_{321} \omega_1 = \omega_1$$

We can summarize our computation by observing that

$$f \cdot W u = (f \otimes u) \cdot W = -f_i u_j \varepsilon_{ijk} \omega_k$$

$$\begin{aligned}
&= -f_1 u_2 \omega_3 + f_2 u_1 \omega_3 \\
&\quad + f_1 u_3 \omega_2 - f_3 u_1 \omega_2 \\
&\quad - f_2 u_3 \omega_1 + f_3 u_2 \omega_1 \\
&= \omega_1 (-f_2 u_3 + f_3 u_2) \\
&\quad + \omega_2 (-f_3 u_1 + f_1 u_3) \\
&\quad + \omega_3 (-f_1 u_2 + f_2 u_1) = u \times f \cdot \omega
\end{aligned}$$

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CROSS PRODUCT DEFINITION

$$u \times f \cdot w = f \otimes u \cdot W \quad \text{with} \quad We = w \times e$$

$$\boxed{u_1 \times u_2 \cdot u_3 = \mathrm{vol}(u_1, u_2, u_3)} \quad \text{definition}$$

for a given vol function

and a given scalar product

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$$v_0 + D(p_A - p_0) = v_0 + D\,u_A$$

$$\begin{aligned}
Du &= D(u_1 e_1 + u_2 e_2 + u_3 e_3) \\
&= u_1(d_{11} e_1 + d_{21} e_2 + d_{31} e_3) \\
&\quad + u_2(d_{12} e_1 + d_{22} e_2 + d_{32} e_3) \\
&\quad + u_3(d_{13} e_1 + d_{32} e_2 + d_{33} e_3) \\
&= (u_1 d_{11} + u_2 d_{12} + u_3 d_{13})\, e_1 \\
&\quad + (u_1 d_{21} + u_2 d_{22} + u_3 d_{23})\, e_2 \\
&\quad + (u_1 d_{31} + u_2 d_{32} + u_3 d_{33})\, e_3
\end{aligned}$$

$$\begin{aligned}
f \cdot Du &= f_1 u_1 d_{11} + f_1 u_2 d_{12} + f_1 u_3 d_{13} \\
&\quad + f_2 u_1 d_{21} + f_2 u_2 d_{22} + f_2 u_3 d_{23} \\
&\quad + f_3 u_1 d_{31} + f_3 u_2 d_{32} + f_3 u_3 d_{33} \\
&= (f \otimes u) \cdot D
\end{aligned}$$

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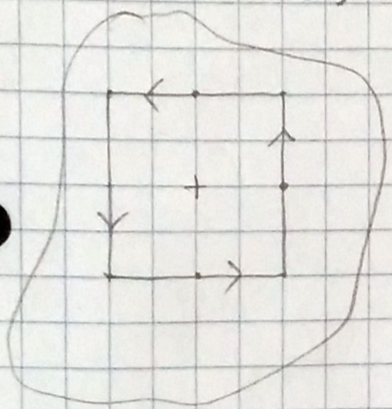
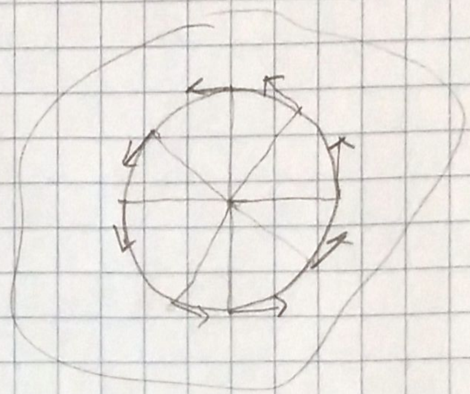
$$L = W + D$$

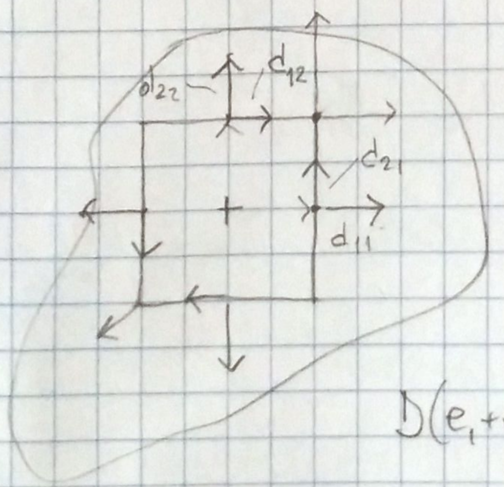
$$v(x) = v(p_0) + L\,(x - p_0)$$

$$v(x) - v(p_0) = L\,(x - p_0)$$



W





$$D e_1 = d_{11} e_1 + d_{21} e_2$$

$$D e_2 = d_{12} e_1 + d_{22} e_2$$

$$D(e_1 + e_2) = D e_1 + D e_2$$
$$= (d_{11} + d_{12}) e_1 + (d_{21} + d_{22}) e_2$$

$$D(e_1 - e_2) = D e_1 - D e_2$$
$$= (d_{11} - d_{12}) e_1 + (d_{21} - d_{22}) e_2$$

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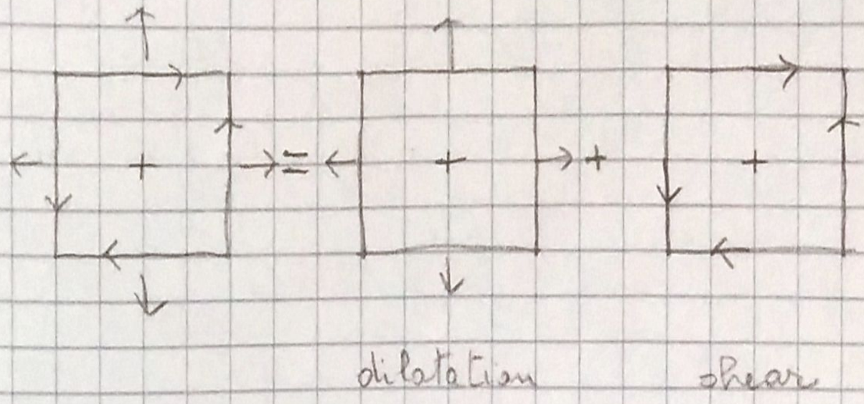

a splitting which is independent of the basis is based on the velocity gradient being an isochoric or a spherical tensor.

For a general tensor, not just the velocity gradient, this decomposition is referred to like the decomposition into spherical and deviatoric parts.

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