BALANCE LAWS

Let us assume that the force distribution is such that

$$\mathfrak{F}(v) = 0 \qquad \forall \text{ test field } v$$

If $\quad v(x) = v_0 + W(x - p_0) \quad$ (rigid velocity fields)

then

for $\quad \mathfrak{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + f_C \cdot v(p_C)$

we get the representation for $\mathfrak{F}$

$$\begin{aligned}\mathfrak{F}(v) &= (f_A + f_B + f_C) \cdot v_0 + \left(f_A \otimes (p_A - p_0) + \dots\right) \cdot W \\ &= f \cdot v_0 + M \cdot W\end{aligned}$$

Hence

$$\mathfrak{F}(v) = 0 \;\; \forall v \quad \Rightarrow \quad f = 0 \,, \;\; \operatorname{skw} M = 0$$

$\mathfrak{F}(v) = 0 \;\; \forall v \quad$ BALANCE LAW

virtual power BALANCE LAW

BALANCE LAW weak form

BALANCE LAW variational form

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If we consider test velocity fields

$$v(x) = v_0 + L\,(x - p_0)$$

Then assume again

$$\mathcal{F}(v) = 0 \qquad \forall v$$

while giving a characterization as regard as the nature of the force distribution

$$\mathcal{F}(v) = \overset{ext}{\mathcal{F}}(v) + \overset{int}{\mathcal{F}}(v)$$

with $\overset{int}{\mathcal{F}}(v) = 0$ $\forall$ rigid test velocity field

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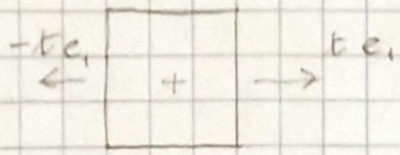
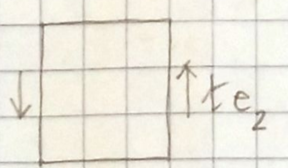


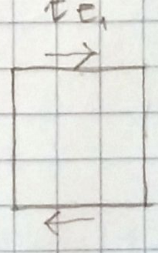
uniform traction on each face

$$\int_{\mathcal{F}_1} t e_1 \otimes \left(\frac{\ell_1}{2} e_1 + s_2 e_2 + s_3 e_3\right) dA$$

$$= \int_{\mathcal{F}_1} t e_1 \otimes \frac{\ell_1}{2} e_1 \, dA + t e_1 \otimes \underbrace{\int_{\mathcal{F}_1} (s_2 e_1 + s_3 e_3)\, dA}_{0}$$

$$M_{11} = t \ell_1 \int_{\mathcal{F}_1} e_1 \otimes e_1 \, dA = t \ell_1 A_{\mathcal{F}_1} \, e_1 \otimes e_1 = t V_{\mathcal{R}} \, e_1 \otimes e_1$$





$$M_{21} = t \ell_1 \int_{\mathcal{F}_1} e_2 \otimes e_1 \, dA \qquad M_{12} = t \ell_2 \int_{\mathcal{F}_2} e_1 \otimes e_2 \, dA$$

$$= t \underbrace{\ell_1 A_{\mathcal{F}_1}}_{V_{\mathcal{R}}} e_2 \otimes e_1 \qquad = t \underbrace{\ell_2 A_{\mathcal{F}_2}}_{V_{\mathcal{R}}} e_1 \otimes e_2$$

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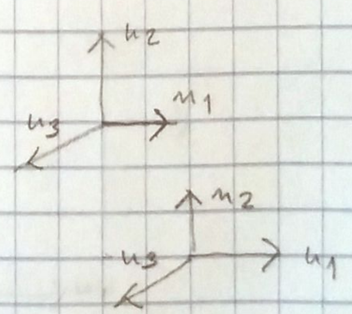
$$M_{21} + M_{12} = t\, V_{\mathcal{R}} \left(e_2 \otimes e_1 + e_1 \otimes e_2 \right)$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{21} - M_{12} = t\, V_{\mathcal{R}} \left(e_2 \otimes e_1 - e_1 \otimes e_2 \right)$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_{\mathcal{F}_1} = \mathrm{vol}(n_1, u_2, u_3)$$



$$A_{\mathcal{F}_2} = \mathrm{vol}(u_1, n_2, u_3)$$

$$A_{\mathcal{F}_3} = \mathrm{vol}(u_1, u_2, n_3)$$

outward unit normal vectors

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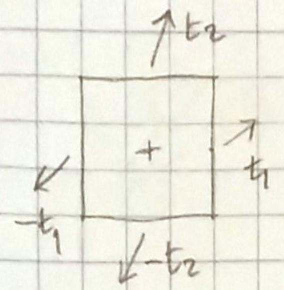

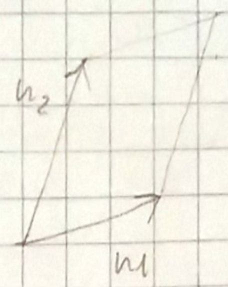

$$t_1 \otimes (x - p_0) = t_1 \otimes \left(x - p_0 - \frac{1}{2} u_1 + \frac{1}{2} u_1\right)$$
$$= t_1 \otimes \left(x - \left(p_0 + \frac{1}{2} u_1\right)\right) + t_1 \otimes \frac{1}{2} u_1$$

$$\int_{\mathcal{F}_1} t_1 \otimes (x - p_0)\, dA = t_1 \otimes \int_{\mathcal{F}_1} (x - p_0)\, dA = t_1 \otimes \frac{1}{2} u_1 A_{\mathcal{F}_1}$$

$$\int_{\mathcal{F}_1} (x - p_0)\, dA = \underbrace{\int_{\mathcal{F}_1}}_{\text{rectangle}} \left(x - \left(p_0 + \frac{1}{2} u_1\right) + \frac{1}{2} u_1\right) dA = \int_{\mathcal{F}_1} \frac{1}{2} u_1\, dA$$

$$t_1 \otimes \frac{1}{2} u_1 A_{\mathcal{F}_1} - t_1 \otimes \left(-\frac{1}{2} u_1\right) A_{\mathcal{F}_1} = t_1 \otimes u_1 A_{\mathcal{F}_1}$$

$$M = (t_1 \otimes u_1) A_{\mathcal{F}_1} + (t_2 \otimes u_2) A_{\mathcal{F}_2} + (t_3 \otimes u_3) A_{\mathcal{F}_3}$$

Note that M turns out to be independent of p_0

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[2016-04-12]

$$M n_1 = A_{\mathcal{F}_1} t_1 (u_1 \cdot n_1) + A_{\mathcal{F}_2} t_2 (u_2 \cdot n_1) + A_{\mathcal{F}_3} t_3 (u_3 \cdot n_1)$$

$$n_1 \cdot u_2 = 0 \qquad n_1 \cdot u_3 = 0 \qquad n_1 \in \operatorname{span}\{u_2, u_3\}^\perp$$

$$w_1 := u_1 - \underbrace{(u_1 \cdot n_1)}_{h_1} n_1 \Rightarrow w_1 \cdot n_1 = 0$$

$$\Rightarrow w_1 \in \operatorname{span}\{u_2, u_3\}$$

$$u_1 = w_1 + h_1 n_1$$

$$V_{\mathcal{R}} = \operatorname{vol}(u_1, u_2, u_3) = \underset{\substack{\downarrow \\ 0}}{\operatorname{vol}(w_1, u_2, u_3)} + h_1 \underbrace{\operatorname{vol}(n_1, u_2, u_3)}_{A_{\mathcal{F}_1}}$$

$$M n_1 = A_{\mathcal{F}_1} t_1 h_1 = V_{\mathcal{R}} t_1$$

$$n_2 \cdot u_3 = 0 \qquad n_2 \cdot u_1 = 0 \qquad n_2 \in \operatorname{span}\{u_3, u_1\}^\perp$$

$$w_2 = u_2 - \underbrace{(u_2 \cdot n_2)}_{h_2} n_2 \Rightarrow w_2 \cdot n_2 = 0$$

$$\Rightarrow w_2 \in \operatorname{span}\{u_3, u_1\}$$

$$u_2 = w_2 + h_2 n_2$$

$$V_{\mathcal{R}} = \operatorname{vol}(u_1, u_2, u_3) = \operatorname{vol}(u_1, w_2, u_3) + h_2 \underbrace{\operatorname{vol}(u_1, n_2, u_3)}_{A_{\mathcal{F}_2}}$$

$$M n_2 = A_{\mathcal{F}_2} t_2 (u_2 \cdot n_2) = V_{\mathcal{R}} t_2$$

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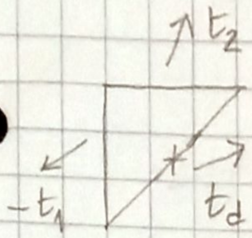
[SKIP]



Half a parallelepiped (prism)

Boundary traction on the diagonal face, from a given moment tensor

$$M = t_1 \otimes \left(\frac{1}{2} u_1\right) A_{\mathcal{F}_1} + t_2 \otimes \left(\frac{1}{2} u_2\right) A_{\mathcal{F}_2}$$

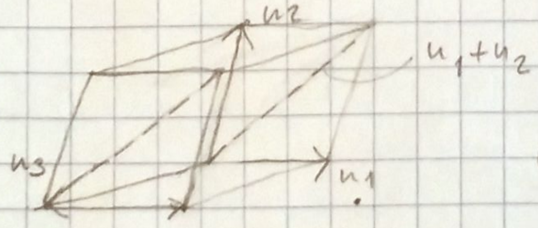
$$+ t_d \otimes (0) A_{\mathcal{F}_d} + t_3 \otimes u_3 \left(\frac{1}{2} A_{\mathcal{F}_3}\right)$$

$$= \frac{1}{2}\left(t_1 \otimes u_1 A_{\mathcal{F}_1} + t_2 \otimes u_2 A_{\mathcal{F}_2} + t_3 \otimes u_3 A_{\mathcal{F}_3}\right)$$

$$M(-n_1) = \frac{1}{2} A_{\mathcal{F}_1} t_1 (-u_1 \cdot n_1) = -\frac{1}{2} A_{\mathcal{F}_1} h_1 t_1 = -\frac{1}{2} V_{\mathcal{R}} t_1$$

$$M n_2 = \dots = \frac{1}{2} V_{\mathcal{R}} t_2$$

$$M n_d = ?$$



$$(u_1 + u_2) \cdot n_d = 0$$

$$u_3 \cdot n_d = 0$$

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[SKIP]

$$\mathrm{vol}(u_1, u_2, u_3) = \mathrm{vol}(u_1+u_2, u_2, u_3) = \mathrm{vol}(u_1, u_2+u_1, u_3)$$

$$n_d \in \mathrm{span}\{u_1+u_2, u_3\}^{\perp}$$

$$w_{d1} = u_1 - \underbrace{(u_1 \cdot n_d)}_{h_{d1}} n_d \qquad w_{d2} = u_2 - \underbrace{(u_2 \cdot n_d)}_{h_{d2}} n_d$$

$$w_{d1} \cdot n_d = 0 \qquad w_{d2} \cdot n_d = 0$$

$$w_{d1}, w_{d2} \in \mathrm{span}\{u_1+u_2, u_3\}$$

$$u_1 = w_{d1} + h_{d1} n_d \qquad u_2 = w_{d2} + h_{d2} n_d$$

$$V_{\mathcal{R}} = \mathrm{vol}(u_1, u_2+u_1, u_3) = h_{d1}\, \mathrm{vol}(n_d, u_2+u_1, u_3) = h_{d1} A_{\mathcal{F}_d}$$

$$V_{\mathcal{R}} = \mathrm{vol}(u_1+u_2, u_2, u_3) = h_{d2}\, \mathrm{vol}(u_1+u_2, n_d, u_3) = -h_{d2} A_{\mathcal{F}_d}$$

$$\Rightarrow h_{d2} = -h_{d1}$$

$$M_{n_d} = \frac{1}{2}\Big(A_{\mathcal{F}_1} t_1 \underbrace{(u_1 \cdot n_d)}_{h_{d1}} + A_{\mathcal{F}_2} t_2 \underbrace{(u_2 \cdot n_d)}_{-h_{d1}} + A_{\mathcal{F}_3} t_3 \underbrace{(u_3 \cdot n_d)}_{0}\Big)$$

$$M_{n_d} = \frac{1}{2} V_{\mathcal{R}} \left(t_1 \frac{A_{\mathcal{F}_1}}{A_{\mathcal{F}_d}} - t_2 \frac{A_{\mathcal{F}_2}}{A_{\mathcal{F}_d}} \right)$$

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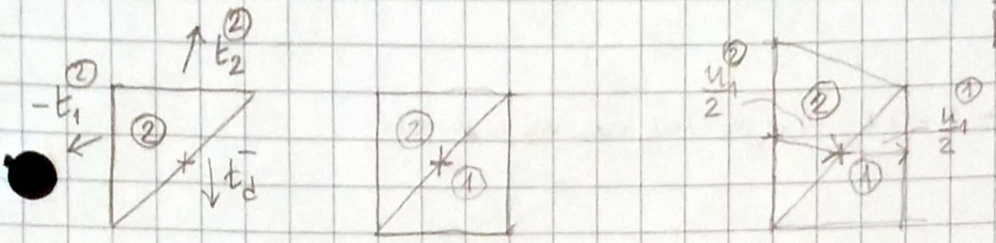
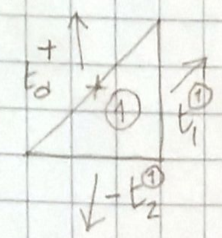
[2016-04-12]





only force densities
per unit volume or
per unit area (boundary)

balance equations

we do not apply forces to interior points, lines, surfaces

$$t_d^{+} A_d^{(1)} + t_1^{(1)} A_{F_1}^{(1)} - t_2^{(1)} A_{F_2}^{(1)} + b^{(1)} V^{(1)} = 0 \qquad (1)$$

$$t_d^{-} A_d^{(2)} - t_1^{(2)} A_{F_1}^{(2)} + t_2^{(2)} A_{F_2}^{(2)} + b^{(2)} V^{(2)} = 0 \qquad (2)$$

$$-t_1^{(2)} A_{F_1}^{(2)} + t_2^{(2)} A_{F_2}^{(2)} - t_2^{(1)} A_{F_2}^{(1)} + t_1^{(1)} A_{F_1}^{(1)} + b^{(1)} V^{(1)} + b^{(2)} V^{(2)} = 0$$

$$\Rightarrow \quad t_d^{+} A_d + t_d^{-} A_d = 0$$

$$\Rightarrow \quad t_d^{+} = - t_d^{-}$$

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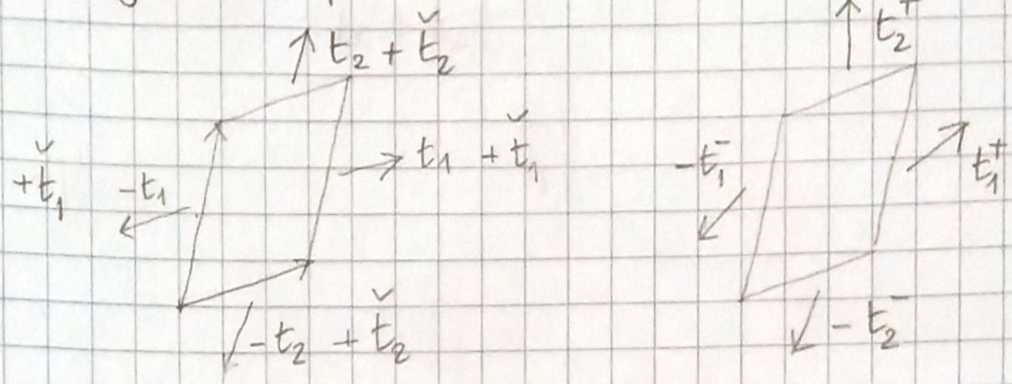
A general force distribution

[2016-04-12]



$$\check{M} = A_{\mathcal{F}_1}\left(\check{t}_1 \otimes \left(\tfrac{1}{2}u_1\right) + \check{t}_1 \otimes \left(-\tfrac{1}{2}u_1\right)\right) + \dots + \dots = 0$$

$$t_1^+ := t_1 + \check{t}_1 \qquad t_2^+ := t_2 + \check{t}_2 \qquad t_3^+ := t_3 + \check{t}_3$$

$$-t_1^- := -t_1 + \check{t}_1 \qquad -t_2^- := -t_2 + \check{t}_2 \qquad -t_3^- := -t_3 + \check{t}_3$$

$$t_1^+ - t_1^- = 2\check{t}_1 \qquad t_2^+ - t_2^- = 2\check{t}_2 \qquad t_3^+ - t_3^- = 2\check{t}_3$$

$$t_1^+ + t_1^- = 2t_1 \qquad t_2^+ + t_2^- = 2t_2 \qquad t_3^+ + t_3^- = 2t_3$$

$$f^{ext} = b\,V_{\mathcal{R}} + (t_1^+ - t_1^-)A_{\mathcal{F}_1} + (t_2^+ - t_2^-)A_{\mathcal{F}_2} + (t_3^+ - t_3^-)A_{\mathcal{F}_3}$$

$$f^{ext} = V_{\mathcal{R}}\left(b + \frac{t_1^+ - t_1^-}{h_1} + \frac{t_2^+ - t_2^-}{h_2} + \frac{t_3^+ - t_3^-}{h_3}\right)$$

We can supplement this expression for f with

$$M^{ext} n_1 = V_{\mathcal{R}}\, t_1 \;;\quad M^{ext} n_2 = V_{\mathcal{R}}\, t_2 \;;\quad M^{ext} n_3 = V_{\mathcal{R}}\, t_3$$

All of the expressions above do not rely on balance equations!

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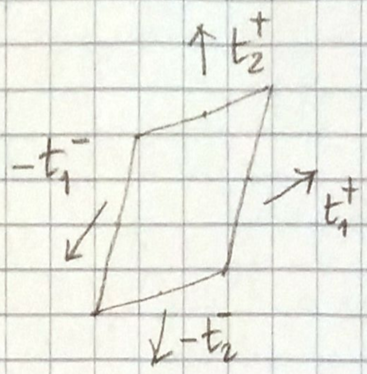
Balance equations

$$\begin{cases} f^{ext} = 0 \\ M^{ext} = T\, V_{\mathcal{R}} \end{cases}$$



$\Rightarrow$

$$\begin{cases} b + \dfrac{t_1^+ - t_1^-}{h_1} + \dfrac{t_2^+ - t_2^-}{h_2} + \dfrac{t_3^+ - t_3^-}{h_3} = 0 \\ \\ T n_1 = t_1 \\ T n_2 = t_2 \\ T n_3 = t_3 \end{cases}$$

with
$$t_1 := \frac{1}{2}\left(t_1^+ + t_1^-\right)$$
$$t_2 := \frac{1}{2}\left(t_2^+ + t_2^-\right)$$
$$t_3 := \frac{1}{2}\left(t_3^+ + t_3^-\right)$$

The same equations can be derived directly from

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial \mathcal{R}} t \cdot v \, dA - V_{\mathcal{R}}\, T \cdot L = 0$$

$$\forall \quad v(x) = v_0 + L(x - p_0)$$

and b a uniform vector field

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$$\mathcal{F}^{ext}(v) = \int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA$$

$$\forall \quad v(x) = v_0 + L(x - p_0)$$

$$\int_{\mathcal{R}} b \cdot v \, dV = \int_{\mathcal{R}} b \cdot v_0 \, dV + \int_{\mathcal{R}} b \cdot L(x - p_0) \, dV$$

$$= V_{\mathcal{R}} \, b \cdot v_0 + \underbrace{\int_{\mathcal{R}} b \otimes (x - p_0) \, dV}_{0} \cdot L = V_{\mathcal{R}} \, b \cdot p_0$$

$$\int_{\partial\mathcal{R}} t \cdot v \, dA = (t_1^+ - t_1^-) \cdot v_0 \, A_{\mathcal{F}_1} + \dots + \dots$$

$$+ \frac{1}{2} A_{\mathcal{F}_1} \left(t_1^+ \otimes u_1 + t_1^- \otimes u_1 \right) \cdot L + \dots + \dots$$

$$= A_{\mathcal{F}_1} (t_1^+ - t_1^-) \cdot v_0 + \dots + \dots$$

$$+ A_{\mathcal{F}_1} \frac{1}{2} (t_1^+ + t_1^-) \otimes u_1 \cdot L + \dots + \dots$$

$$f^{ext} = b \, V_{\mathcal{R}} + A_{\mathcal{F}_1} (t_1^+ - t_1^-) + A_{\mathcal{F}_2} (t_2^+ - t_2^-) + A_{\mathcal{F}_3} (t_3^+ - t_3^-)$$

$$M^{ext} = \frac{1}{2} A_{\mathcal{F}_1} (t_1^+ + t_1^-) \otimes u_1 + \frac{1}{2} A_{\mathcal{F}_2} (t_2^+ + t_2^-) \otimes u_2 + \frac{1}{2} A_{\mathcal{F}_3} (t_3^+ + t_3^-) \otimes u_3$$

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We get from f^{ext} a volume density

$$\frac{1}{V_R} f^{ext} = b + \frac{A_{F_1}}{V_R}(t_1^+ - t_1^-) + \frac{A_{F_2}}{V_R}(t_2^+ - t_1^-) + \frac{A_{F_3}}{V_R}(t_3^+ - t_1^-)$$

thus applying sort of a "divergence theorem" transforming an "area integral" into a "volume integral"

$$* \quad \frac{1}{V_R} f^{ext} = b + \frac{t_1^+ - t_1^-}{h_1} + \frac{t_2^+ - t_2^-}{h_2} + \frac{t_3^+ - t_3^-}{h_3}$$

We get a volume density from M^{ext} as well

$$\frac{1}{V_R} M^{ext} = \frac{A_{F_1}}{V_R}\frac{t_1^+ + t_1^-}{2} \otimes n_1 + \frac{A_{F_2}}{V_R}\frac{t_2^+ + t_2^-}{2} \otimes n_2 + \frac{A_{F_3}}{V_R}\frac{t_3^+ + t_3^-}{2} \otimes n_3$$

* which can equivalently be described by using the basis $\{n_1, n_2, n_3\}$

$$\frac{1}{V_R} M^{ext} n_1 = \frac{t_1^+ + t_1^-}{2}$$

$$\frac{1}{V_R} M^{ext} n_2 = \frac{t_2^+ + t_2^-}{2}$$

$$\frac{1}{V_R} M^{ext} n_3 = \frac{t_3^+ + t_3^-}{2}$$

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The boundary traction can be split into two parts

$$\mathcal{F}_\partial(v) = f_\partial \cdot v_o + M_\partial \cdot L$$

The second part (whose total force is zero) is described by M, whose symmetric part can be balanced by T.
The first part (whose total moment is zero) cannot be balanced by any stress.
It can only be balanced by the bulk force.

$$\mathcal{F}^{ext}(v) = \mathcal{F}_b(v) + \mathcal{F}_\partial(v)$$

BULK BOUNDARY

$$(f_b + f_\partial)\cdot v_o + (M_b + M_\partial)\cdot L$$

$$b V_\mathcal{R} \quad \check{t} A_{\partial\mathcal{R}} \qquad b V_\mathcal{R} \ell \quad t A_{\partial\mathcal{R}} \ell$$

$$V_\mathcal{R}(b, \check{t}/h) \qquad V_\mathcal{R}(b\ell, t)$$

*** $\lim_{h\to 0} \check{t} = 0$ in order for $\lim_{h\to 0} \check{t}/h$ not to grow to infinity

$\lim_{h\to 0} f_\partial / V_\mathcal{R} < \infty$, $\lim_{h\to 0} M_b / V_\mathcal{R} = 0$

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The limit process should refer to a sequence generated this way:

let us consider a finite partition of the shape $\mathcal{R}$ (volume $V_{\mathcal{R}}$) into n tetrahedra $\mathcal{T}_i$;

then let $n \to \infty$ while $V_{\mathcal{T}_i} \to 0 \;\; \forall \mathcal{T}_i$

in a uniform way:

for any $V_{\tau} < V_{\mathcal{R}}$ there is a partition in the sequence above where $V_{\tau_i} < V_{\tau} \quad \forall \mathcal{T}_i$

(the same uniformity condition should be fulfilled by the area of any face and by the lenght of any edge)

We should get $\lim \check{t} = 0$ in order for

$$\lim f_o / V_{\tau} < \infty \qquad \forall \mathcal{T}_i$$

together with

$$\lim M_o / V_{\tau} = 0 \qquad \forall \mathcal{T}_i$$

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