[2016-05-12]

Energy imbalance principle

Force balance

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \qquad \forall v \text{ test}$$

Power expended in any motion

$$\int_{\mathcal{R}} b\cdot v \, dV + \int_{\partial\mathcal{R}} t\cdot v \, dA = \int_{\mathcal{R}} T\cdot \dot{F}F^{-1} \, dV$$

$$\int_{\mathcal{R}} T\cdot \dot{F}F^{-1} \, dV = \int_{\bar{\mathcal{R}}} S\cdot \dot{F} \, dV$$

energy imbalance principle

$$S\cdot\dot{F} - \underbrace{\frac{d}{dt}\varphi(F)}_{\hat{S}(F)\cdot\dot{F}} \geq 0 \qquad \text{dissipation inequality}$$

$$\left(S - \hat{S}(F)\right)\cdot\dot{F} \geq 0 \qquad \forall \text{ motion}$$

$$\cancel{\underset{>0}{(\det F)}}\,\big(\underbrace{T - \hat{T}(F)}_{T^{+}}\big)\cdot\dot{F}F^{-T} \geq 0 \qquad \forall \text{ motion}$$

T^{+} ← dissipative part of the stress

$$T = \hat{T}(F) + T^{+}$$

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Characterizing the dissipation mechanism

T^+ cannot be a constant tensor

It should depend on the velocity gradient
in order to fulfill the inequality

$$T^+ \cdot \dot{F}F^{-1} \geq 0$$

The simplest choice is

$$T^+ = 2\,\mu \operatorname{sym} \dot{F}F^{-1}$$

$$\mu\, \dot{F}F^{-1} \cdot \dot{F}F^{-1} \geq 0 \quad \Leftrightarrow \mu \geq 0$$

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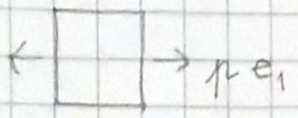

$$T = \hat{T}_e(F) - pI + 2\mu\, \mathrm{sym}(\dot{F}F^{-1})$$

$$\sigma_1 = \hat{\sigma}_1(\lambda) - p + 2\mu \frac{\dot\lambda}{\lambda} \qquad \sigma_1 = \mathfrak{p}$$

$$\sigma_2 = \hat{\sigma}_2(\lambda) - p - \mu \frac{\dot\lambda}{\lambda} \qquad \sigma_2 = 0$$

$$\sigma_3 = \hat{\sigma}_3(\lambda) - p - \mu \frac{\dot\lambda}{\lambda} \qquad \sigma_3 = 0$$

(material response) (balance equations)

$\mathrm{tr}\, T_e(F) = 0$

$$\hat{\sigma}_1(\lambda) - p + 2\mu \frac{\dot\lambda}{\lambda} = \mathfrak{p}$$

$$\hat{\sigma}_2(\lambda) - p - \mu \frac{\dot\lambda}{\lambda} = 0$$

$$\hat{\sigma}_3(\lambda) - p - \mu \frac{\dot\lambda}{\lambda} = 0$$

taking the trace

$$0 - 3p + 0 = \mathfrak{p}$$

$$\Rightarrow \quad p = -\frac{1}{3}\mathfrak{p}$$

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$$\hat{\sigma}_1(\lambda) = \frac{4}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\frac{4}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right) + \frac{1}{3}p + 2\mu\frac{\dot{\lambda}}{\lambda} = p$$

$$\hat{\sigma}_2(\lambda) - \hat{\sigma}_3(\lambda) = 0$$

$$\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) - 2p - 2\mu\frac{\dot{\lambda}}{\lambda} = 0$$

$$\hat{\sigma}_2(\lambda) + \hat{\sigma}_3(\lambda) = -\hat{\sigma}_1(\lambda) = -\frac{4}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\Rightarrow \quad \hat{\sigma}_2(\lambda) = -\frac{2}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\hat{\sigma}_3(\lambda) = -\frac{2}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\frac{4}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right) + 2\mu\frac{\dot{\lambda}}{\lambda} = \frac{2}{3}p$$

$$\boxed{2c\left(\lambda^2 - \frac{1}{\lambda}\right) + 3\mu\frac{\dot{\lambda}}{\lambda} = p}$$

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Equivalently

$$\hat{T}_e(F) = 2c_1\left(FF^T - \frac{1}{3}\operatorname{tr}(FF^T)I\right) \quad \text{neo-Hookean}$$

$$[FF^T] = \begin{pmatrix} \lambda^2 & & \\ & 1/\lambda & \\ & & 1/\lambda \end{pmatrix} \qquad \dot{F}F^{-1} = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix}\frac{\dot{\lambda}}{\lambda}$$

$$[\hat{T}_e(F)] = 2c_1\begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\hat{T}_e(F) - pI + 2\mu \operatorname{sym}\dot{F}F^{-1} = \pi\, e_1 \otimes e_1$$

(balance & material characterization)

sph $\qquad -p\,3 = \pi$

der $\qquad \hat{T}_e(F) + 2\mu \operatorname{sym}\dot{F}F^{-1} = \pi\left(e_1\otimes e_1 - \frac{1}{3}I\right)$

$$\frac{4}{3}c\left(\lambda^2 - \frac{1}{\lambda}\right) + 2\mu\frac{\dot{\lambda}}{\lambda} = \frac{2}{3}\pi$$

$$\boxed{2c\left(\lambda^2 - \frac{1}{\lambda}\right) + 3\mu\frac{\dot{\lambda}}{\lambda} = \pi}$$

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linearization around λ_0 (stationary solution)

such that

$$2c\left(\lambda_0^2 - \frac{1}{\lambda_0}\right) = p$$

$$2c\left(2\lambda_0 + \frac{1}{\lambda_0^2}\right)\underbrace{(\lambda-\lambda_0)}_{\tilde{\lambda}} + 3\mu\frac{\dot{\lambda}}{\lambda_0} = 0$$

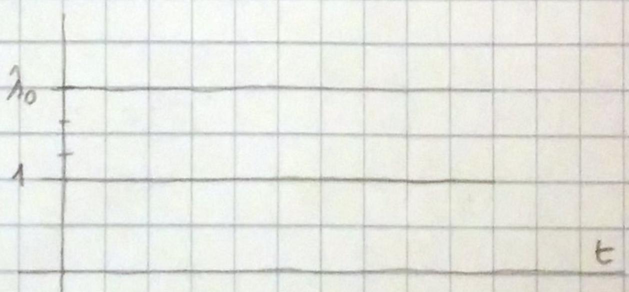
$$\frac{2}{3}c\left(2\lambda_0^2+\frac{1}{\lambda_0}\right)(\lambda-\lambda_0) = -\mu\dot{\lambda}$$

$$\lambda(t) = \lambda_0 + \tilde{\lambda}_0 e^{-\frac{\alpha}{\mu}t} \;; \qquad \tilde{\lambda}(t) = \tilde{\lambda}_0 e^{-\frac{\alpha}{\mu}t}$$

$$\frac{2}{3}c\left(2\lambda_0^2+\frac{1}{\lambda_0}\right)\tilde{\lambda}_0 e^{-\frac{\alpha}{\mu}t} = +\mu\frac{\alpha}{\mu}e^{-\frac{\alpha}{\mu}t}$$

$$\Rightarrow \quad \frac{2}{3}c\left(2\lambda_0+\frac{1}{\lambda_0}\right) = \alpha \qquad \Rightarrow \quad \alpha > 0$$

$t=0 \qquad \lambda(0) - \lambda_0 = \tilde{\lambda}_0$



$\lambda(0) = 1 \quad \Rightarrow \quad 1-\lambda_0 = \tilde{\lambda}_0$ & $(1-\lambda_0)$ small

$$\Rightarrow \quad \lambda(t) = \lambda_0 + \tilde{\lambda}_0 e^{-\frac{\alpha}{\mu}t} = \lambda_0 + (1-\lambda_0)e^{-\frac{\alpha}{\mu}t}$$

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