Placements and deformations

In distinguishing the kinds of physical quantities, it is of great importance to know how they are related to the directions of those coordinate axes which we usually employ in defining the positions of things. The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. [...]

But for many purpose of physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other [...].

[Maxwell J. C., *A Treatise on Electricity and Magnetism*, Oxford University Press, 1892, vol. I, p. 9]

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1 Placement and deformation

Denoting by $\mathcal{B}$ a set $\{\mathsf{A}, \mathsf{B}, \dots\}$ made up of *body points*, a *placement* is a map

$$\mathsf{p} : \mathcal{B} \to \mathcal{E} \tag{1}$$

which assigns a position to each point, in such a way that different body points (or simply *points*) take different positions. The *shape* of the body $\mathcal{B}$ is the set

$$\mathcal{R} := \operatorname{im} \mathsf{p}. \tag{2}$$

We call *configuration* of $\mathcal{B}$ the collection of the couples

$$\big(\mathsf{A}, \mathsf{p}(\mathsf{A})\big) \quad \forall \mathsf{A} \in \mathcal{B}\,, \tag{3}$$

where p is a *placement*. For any two placements $\bar{\mathsf{p}}$ and p, corresponding to shapes $\bar{\mathcal{R}}$ and $\mathcal{R}$ there exists a bijective map, called *deformation*,

$$\boldsymbol{\phi} : \bar{\mathcal{R}} \to \mathcal{R}, \tag{4}$$

defined as $\boldsymbol{\phi} := \mathsf{p} \circ (\bar{\mathsf{p}})^{-1}$, which moves every point $\mathsf{A} \in \mathcal{B}$ from position $\bar{\mathsf{p}}(\mathsf{A})$ to position

$$\mathsf{p}(\mathsf{A}) = \boldsymbol{\phi}(\bar{\mathsf{p}}(\mathsf{A})). \tag{5}$$

We can define also the *displacement field*,

$$\mathbf{u} : \bar{\mathsf{p}}(\mathsf{A}) \mapsto (\mathsf{p}(\mathsf{A}) - \bar{\mathsf{p}}(\mathsf{A})) \quad \forall \mathsf{A} \in \mathcal{B}. \tag{6}$$

2 Rigid deformations and affine deformations

If the deformation $\boldsymbol{\phi}$ is an isometry which leaves the orientation unchanged we call it a *rigid deformation*. In a rigid deformation, however we choose a body point A the position of any other body point B in the placement p is given by the following expression

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{B}}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{R}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}), \tag{7}$$

where $\mathbf{R}$ is a rotation of $\mathcal{V}$.

If $\boldsymbol{\phi}$ is an affine map which leaves the orientation unchanged then we call it an *affine deformation* or a *homogenous deformation*. However we choose a body point A the position of any other body point B in the placement p is given by the following expression

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{B}}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{F}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}), \tag{8}$$

where $\mathbf{F}$ is an endomorphism of $\mathcal{V}$, such that $\det \mathbf{F} > 0$. Note that the line

$$\bar{\boldsymbol{c}}(h) = \bar{\mathsf{p}}_{\mathsf{A}} + h\, \bar{\mathbf{a}} \tag{9}$$

is transformed, through (8), into the curve

$$\boldsymbol{c}(h) := \boldsymbol{\phi}(\bar{\boldsymbol{c}}(h)) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{F}(\bar{\boldsymbol{c}}(h) - \bar{\mathsf{p}}_{\mathsf{A}}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + h\, \mathbf{F}\bar{\mathbf{a}}. \tag{10}$$

This curve is nothing but the line

$$\boldsymbol{c}(h) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + h\, \mathbf{a} \tag{11}$$

whose tangent vector is

$$\mathbf{a} = \mathbf{F}\bar{\mathbf{a}}. \tag{12}$$

Hence an affine deformation transforms lines into lines.

3 Composition of deformations

Let us consider the following composition of two affine deformations

$$\boldsymbol{\phi} := \boldsymbol{\phi}_{[2]}\boldsymbol{\phi}_{[1]}. \tag{13}$$

For any two body points A and B, the first deformation is such that

$$\boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{B}}) = \boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{F}_{[1]}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}). \tag{14}$$

Applying the second deformation we get

$$\boldsymbol{\phi}_{[2]}(\boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{B}})) = \boldsymbol{\phi}_{[2]}(\boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{A}})) + \mathbf{F}_{[2]}(\boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{B}}) - \boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{A}})) \tag{15}$$

and, by substituting (14),

$$\boldsymbol{\phi}_{[2]}(\boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{B}})) = \boldsymbol{\phi}_{[2]}(\boldsymbol{\phi}_{[1]}(\bar{\mathsf{p}}_{\mathsf{A}})) + \mathbf{F}_{[2]}(\mathbf{F}_{[1]}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}})). \tag{16}$$

Hence for the composition (13) is described by

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{B}}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{F}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}) \tag{17}$$

where

$$\mathbf{F} = \mathbf{F}_{[2]}\mathbf{F}_{[1]}. \tag{18}$$

Note that in general neither composition (13) nor composition (18) are commutative.

By using the polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \tag{19}$$

every affine deformation $\boldsymbol{\phi}$ can be expressed, after choosing any body point A, as

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{B}}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{R}\mathbf{U}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}) \tag{20}$$

and then decomposed into a translation

$$\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{B}}) = \boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{A}}) + (\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}), \tag{21}$$

such that $\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{A}}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}})$, followed by a stretch, while holding $\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}})$ fixed,

$$\boldsymbol{\phi}_{[1]}(\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{B}})) = \boldsymbol{\phi}_{[1]}(\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{A}})) + \mathbf{U}(\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{B}}) - \boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{A}})) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{U}(\bar{\mathsf{p}}_{\mathsf{B}} - \bar{\mathsf{p}}_{\mathsf{A}}), \tag{22}$$

followed in turn by a rotation, with center $\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}})$,

$$\boldsymbol{\phi}_{[2]}(\boldsymbol{\phi}_{[1]}(\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{B}}))) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}) + \mathbf{R}(\boldsymbol{\phi}_{[1]}(\boldsymbol{\phi}_{[0]}(\bar{\mathsf{p}}_{\mathsf{B}})) - \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}})). \tag{23}$$

The orthogonal lines through $\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}})$ generated by the eigenvectors of $\mathbf{U}$, called the *principal directions* of the stretch, are invariant under the deformation $\boldsymbol{\phi}_{[1]}$. The distance between any two positions along the principal directions changes by a factor equal to the corresponding eigenvalue of $\mathbf{U}$, called the *principal stretches*. The line through $\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}})$ generated by the eigenvector of $\mathbf{R}$, called *rotation axis*, is invariant under the deformation $\boldsymbol{\phi}_{[2]}$.

4 Affine deformations in coordinate form

Given a Cartesian coordinate system in a Euclidean space $\mathcal{E}$ of dimension two, the positions of any two body points $\mathsf{A}, \mathsf{B} \in \mathcal{B}$ in the placement $\bar{\mathsf{p}}$ can be described by the expressions

$$\begin{aligned} \bar{\mathsf{p}}_\mathsf{A} &= o + \bar{x}_{1\mathsf{A}} \mathbf{e}_1 + \bar{x}_{2\mathsf{A}} \mathbf{e}_2, \\ \bar{\mathsf{p}}_\mathsf{B} &= o + \bar{x}_{1\mathsf{B}} \mathbf{e}_1 + \bar{x}_{2\mathsf{B}} \mathbf{e}_2. \end{aligned} \tag{24}$$

The positions of the same two body points in the placement p can be described by the expressions

$$\begin{aligned} \boldsymbol{\phi}(\bar{\mathsf{p}}_\mathsf{A}) &= o + x_{1\mathsf{A}} \mathbf{e}_1 + x_{2\mathsf{A}} \mathbf{e}_2, \\ \boldsymbol{\phi}(\bar{\mathsf{p}}_\mathsf{B}) &= o + x_{1\mathsf{B}} \mathbf{e}_1 + x_{2\mathsf{B}} \mathbf{e}_2. \end{aligned} \tag{25}$$

For a rigid deformation, by substituting (24) and (25) into (7), we get

$$o + x_{1\mathsf{B}} \mathbf{e}_1 + x_{2\mathsf{B}} \mathbf{e}_2 = o + x_{1\mathsf{A}} \mathbf{e}_1 + x_{2\mathsf{A}} \mathbf{e}_2 + \mathbf{R}\big((\bar{x}_{1\mathsf{B}} - \bar{x}_{1\mathsf{A}})\mathbf{e}_1 + (\bar{x}_{2\mathsf{B}} - \bar{x}_{2\mathsf{A}})\mathbf{e}_2\big), \tag{26}$$

which implies

$$x_{1\mathsf{B}} \mathbf{e}_1 + x_{2\mathsf{B}} \mathbf{e}_2 = x_{1\mathsf{A}} \mathbf{e}_1 + x_{2\mathsf{A}} \mathbf{e}_2 + (\bar{x}_{1\mathsf{B}} - \bar{x}_{1\mathsf{A}})\mathbf{R}\mathbf{e}_1 + (\bar{x}_{2\mathsf{B}} - \bar{x}_{2\mathsf{A}})\mathbf{R}\mathbf{e}_2. \tag{27}$$

We can parameterize the rotation tensor through an angle θ in the following way

$$\mathbf{R}\mathbf{e}_1 = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2, \tag{28}$$

$$\mathbf{R}\mathbf{e}_2 = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2. \tag{29}$$

Then, by equating components in the vector equation (27) we obtain the coordinate description of the rigid deformation

$$\begin{pmatrix} x_{1\mathsf{B}} \\ x_{2\mathsf{B}} \end{pmatrix} = \begin{pmatrix} x_{1\mathsf{A}} \\ x_{2\mathsf{A}} \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{x}_{1\mathsf{B}} - \bar{x}_{1\mathsf{A}} \\ \bar{x}_{2\mathsf{B}} - \bar{x}_{2\mathsf{A}} \end{pmatrix}. \tag{30}$$

For an affine deformation, by substituting (24) and (25) into (8) and setting

$$\mathbf{F}\mathbf{e}_1 = f_{11} \mathbf{e}_1 + f_{21} \mathbf{e}_2, \tag{31}$$

$$\mathbf{F}\mathbf{e}_2 = f_{12} \mathbf{e}_1 + f_{22} \mathbf{e}_2, \tag{32}$$

we obtain the following coordinate description

$$\begin{pmatrix} x_{1\mathsf{B}} \\ x_{2\mathsf{B}} \end{pmatrix} = \begin{pmatrix} x_{1\mathsf{A}} \\ x_{2\mathsf{A}} \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \bar{x}_{1\mathsf{B}} - \bar{x}_{1\mathsf{A}} \\ \bar{x}_{2\mathsf{B}} - \bar{x}_{2\mathsf{A}} \end{pmatrix}. \tag{33}$$

5 Parameterizations

Given a Cartesian coordinate system in a Euclidean space $\mathcal{E}$ of dimension two, the position of a body point A in the placement $\bar{\mathsf{p}}$ is described by (24). If we set $s_{1\mathsf{A}} := \bar{x}_{1\mathsf{A}}$, $s_{2\mathsf{A}} := \bar{x}_{2\mathsf{A}}$, then (24) becomes

$$\bar{\mathsf{p}}_\mathsf{A} = o + s_{1\mathsf{A}} \mathbf{e}_1 + s_{2\mathsf{A}} \mathbf{e}_2. \tag{34}$$

Hence, for each position there exists a couple of coordinates

$$\bar{\mathsf{p}}_\mathsf{A} \mapsto (s_{1\mathsf{A}}, s_{2\mathsf{A}}) \tag{35}$$

and viceversa any couple of coordinates defines a position

$$(s_{1A}, s_{2A}) \mapsto \bar{\mathsf{p}}_A. \tag{36}$$

Denoting by $\boldsymbol{\kappa}$ such a function, called *parameterization* of the body shape $\bar{\mathcal{R}} := \operatorname{im} \bar{\mathsf{p}}$, we get

$$\bar{\mathsf{p}}_A = \boldsymbol{\kappa}(s_{1A}, s_{2A}). \tag{37}$$

In general the parameterization of a body shape is independent of the coordinate system. If the dimension of the Euclidean space is 2, we assume that the domain of the parameterization is the closure of an open set of $\mathbb{R}^2$. Since the deformation gradient $\mathbf{F}$ is never singular the dimension of the shape does not change.

A coordinate description of a deformation $\boldsymbol{\phi}$ can be given through two scalar functions ϕ_1 and ϕ_2 such that

$$\boldsymbol{\phi}(\boldsymbol{\kappa}(s_1, s_2)) = \boldsymbol{o} + \phi_1(s_1, s_2)\mathbf{e}_1 + \phi_2(s_1, s_2)\mathbf{e}_2. \tag{38}$$

If the dimension of the Euclidean space is 3, we assume that the domain of the parameterization is the closure of an open set of $\mathbb{R}^3$. In this case we need three scalar functions ϕ_1, ϕ_2 and ϕ_3 in order to give a coordinate description of the deformation.

6 Deformation gradient

Let us consider a placement $\bar{\mathsf{p}}$ and the line through $\bar{\mathsf{p}}_O = \boldsymbol{\kappa}(s_1, s_2)$

$$\bar{c}_1(h) := \bar{\mathsf{p}}_O + h\,\mathbf{e}_1. \tag{39}$$

By using a coordinate system as in (34) and the parameterization (37) we obtain the following description

$$\bar{c}_1(h) := \bar{\mathsf{p}}_O + h\,\mathbf{e}_1 = \boldsymbol{\kappa}(s_1, s_2) + h\,\mathbf{e}_1 = \boldsymbol{o} + (s_1 + h)\mathbf{e}_1 + s_2\mathbf{e}_2 = \boldsymbol{\kappa}(s_1 + h, s_2). \tag{40}$$

This line is transformed by $\boldsymbol{\phi}$ into the curve

$$\boldsymbol{c}_1(h) := \boldsymbol{\phi}(\bar{c}_1(h)) = \boldsymbol{\phi}(\boldsymbol{\kappa}(s_1 + h, s_2)). \tag{41}$$

If the deformation $\boldsymbol{\phi}$ is not affine then this curve is not a straight line in general. By using the coordinate expression (38) for $\boldsymbol{\phi}$, we get the following coordinate expression for the curve (41)

$$\boldsymbol{c}_1(h) = \boldsymbol{o} + \phi_1(s_1 + h, s_2)\mathbf{e}_1 + \phi_2(s_1 + h, s_2)\mathbf{e}_2, \tag{42}$$

$$\boldsymbol{c}_1(0) = \boldsymbol{o} + \phi_1(s_1, s_2)\mathbf{e}_1 + \phi_2(s_1, s_2)\mathbf{e}_2. \tag{43}$$

Hence the tangent vector at $\boldsymbol{\phi}(\bar{\mathsf{p}}_O)$ is given by

$$\boldsymbol{c}_1' = \lim_{h \to 0} \frac{1}{h}\left(\boldsymbol{c}_1(h) - \boldsymbol{c}_1(0)\right) = \partial_1\phi_1\,\mathbf{e}_1 + \partial_1\phi_2\,\mathbf{e}_2, \tag{44}$$

where ∂_1 is the derivative with respect to the first argument. The line

$$\bar{c}_2(h) := \bar{\mathsf{p}}_O + h\,\mathbf{e}_2 = \boldsymbol{\kappa}(s_1, s_2) + h\,\mathbf{e}_2 = \boldsymbol{o} + s_1\mathbf{e}_1 + (s_2 + h)\mathbf{e}_2 = \boldsymbol{\kappa}(s_1, s_2 + h) \tag{45}$$

is transformed into the curve

$$\boldsymbol{c}_2(h) := \boldsymbol{\phi}(\bar{c}_2(h)) = \boldsymbol{\phi}(\boldsymbol{\kappa}(s_1, s_2 + h)). \tag{46}$$

By using (38) we get

$$\boldsymbol{c}_2(h) = \boldsymbol{o} + \phi_1(s_1, s_2 + h)\mathbf{e}_1 + \phi_2(s_1, s_2 + h)\mathbf{e}_2, \tag{47}$$
$$\boldsymbol{c}_2(0) = \boldsymbol{o} + \phi_1(s_1, s_2)\mathbf{e}_1 + \phi_2(s_1, s_2)\mathbf{e}_2. \tag{48}$$

hence the tangent vector at $\boldsymbol{\phi}(\bar{\mathsf{p}}_O)$ is given by

$$\boldsymbol{c}_2' = \lim_{h\to 0} \frac{1}{h}\left(\boldsymbol{c}_2(h) - \boldsymbol{c}_2(0)\right) = \partial_2\phi_1\,\mathbf{e}_1 + \partial_2\phi_2\,\mathbf{e}_2, \tag{49}$$

where ∂_2 is the derivative with respect to the second argument. Finally, the line

$$\begin{aligned}\bar{\boldsymbol{c}}(h) := \bar{\mathsf{p}}_O + h\,\bar{\mathbf{a}} &= \boldsymbol{\kappa}(s_1, s_2) + h\,\bar{\mathbf{a}} = \boldsymbol{\kappa}(s_1, s_2) + h(\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2)\\ &= \boldsymbol{o} + (s_1 + h\alpha_1)\mathbf{e}_1 + (s_2 + h\alpha_2)\mathbf{e}_2 = \boldsymbol{\kappa}(s_1 + h\alpha_1, s_2 + h\alpha_2)\end{aligned} \tag{50}$$

is transformed into the curve

$$\boldsymbol{c}(h) := \boldsymbol{\phi}(\bar{\boldsymbol{c}}(h)) = \boldsymbol{\phi}(\boldsymbol{\kappa}(s_1 + h\alpha_1, s_2 + h\alpha_2)). \tag{51}$$

Again, from (38) we obtain

$$\boldsymbol{c}(h) = \boldsymbol{o} + \phi_1(s_1 + h\alpha_1, s_2 + h\alpha_2)\mathbf{e}_1 + \phi_2(s_1 + h\alpha_1, s_2 + h\alpha_2)\mathbf{e}_2, \tag{52}$$
$$\boldsymbol{c}(0) = \boldsymbol{o} + \phi_1(s_1, s_2)\mathbf{e}_1 + \phi_2(s_1, s_2)\mathbf{e}_2. \tag{53}$$

Thus the tangent vector at $\boldsymbol{\phi}(\bar{\mathsf{p}}_O)$ turns out to be

$$\begin{aligned}\boldsymbol{c}' &= \lim_{h\to 0}\frac{1}{h}\left(\boldsymbol{c}(h) - \boldsymbol{c}(0)\right) = \left(\alpha_1\,\partial_1\phi_1 + \alpha_2\,\partial_2\phi_1\right)\mathbf{e}_1 + \left(\alpha_1\,\partial_1\phi_2 + \alpha_2\,\partial_2\phi_2\right)\mathbf{e}_2\\ &= \alpha_1\left(\partial_1\phi_1\,\mathbf{e}_1 + \partial_1\phi_2\,\mathbf{e}_2\right) + \alpha_2\left(\partial_2\phi_1\,\mathbf{e}_1 + \partial_2\phi_2\,\mathbf{e}_2\right).\end{aligned} \tag{54}$$

Note that

$$\bar{\boldsymbol{c}}_1' = \lim_{h\to 0}\frac{1}{h}\left(\bar{\boldsymbol{c}}_1(h) - \bar{\boldsymbol{c}}_1(0)\right) = \mathbf{e}_1, \tag{55}$$
$$\bar{\boldsymbol{c}}_2' = \lim_{h\to 0}\frac{1}{h}\left(\bar{\boldsymbol{c}}_2(h) - \bar{\boldsymbol{c}}_2(0)\right) = \mathbf{e}_2, \tag{56}$$
$$\bar{\boldsymbol{c}}' = \lim_{h\to 0}\frac{1}{h}\left(\bar{\boldsymbol{c}}(h) - \bar{\boldsymbol{c}}(0)\right) = \bar{\mathbf{a}} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2. \tag{57}$$

It follows that tangent vectors are such that

$$\bar{\boldsymbol{c}}' = \alpha_1\,\bar{\boldsymbol{c}}_1' + \alpha_2\,\bar{\boldsymbol{c}}_2', \tag{58}$$
$$\boldsymbol{c}' = \alpha_1\,\boldsymbol{c}_1' + \alpha_2\,\boldsymbol{c}_2', \tag{59}$$

Then there exists a linear map, called the *deformation gradient* $\mathbf{F}(\bar{\mathsf{p}}_O)$ transforming tangent vectors to lines at $\bar{\mathsf{p}}_O$ into tangent vectors to corresponding curves at $\boldsymbol{\phi}(\bar{\mathsf{p}}_O)$:

$$\mathbf{F}(\bar{\mathsf{p}}_O) : \bar{\boldsymbol{c}}_1' \mapsto \boldsymbol{c}_1', \tag{60}$$
$$\mathbf{F}(\bar{\mathsf{p}}_O) : \bar{\boldsymbol{c}}_2' \mapsto \boldsymbol{c}_2', \tag{61}$$
$$\mathbf{F}(\bar{\mathsf{p}}_O) : \alpha_1\,\bar{\boldsymbol{c}}_1' + \alpha_2\,\bar{\boldsymbol{c}}_2' \mapsto \alpha_1\,\boldsymbol{c}_1' + \alpha_2\,\boldsymbol{c}_2'. \tag{62}$$

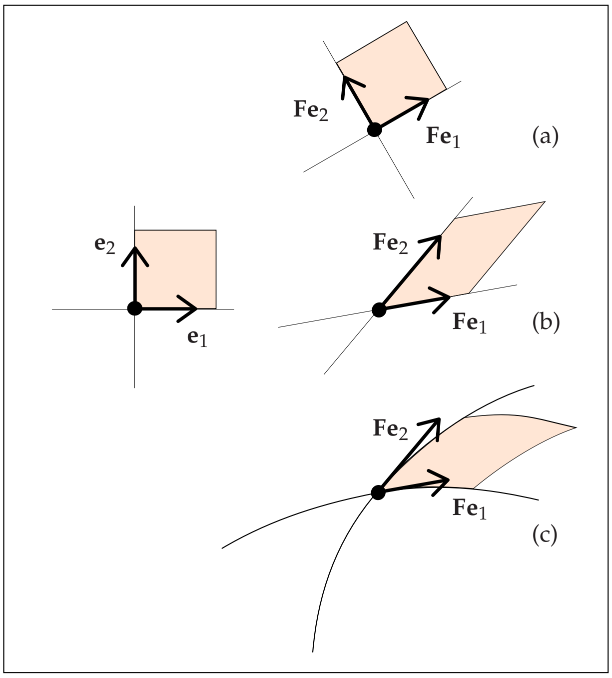

Figure 1: Deformation: (a) rigid, (b) affine, (c) generic.

From the expressions for $\boldsymbol{c}_1'$ and $\boldsymbol{c}_2'$ we obtain

$$\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O})\mathbf{e}_1 = \partial_1\phi_1\,\mathbf{e}_1 + \partial_1\phi_2\,\mathbf{e}_2, \tag{63}$$

$$\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O})\mathbf{e}_2 = \partial_2\phi_1\,\mathbf{e}_1 + \partial_2\phi_2\,\mathbf{e}_2. \tag{64}$$

Hence the matrix of $\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O})$ is

$$[\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O})] = \begin{pmatrix} \dfrac{\partial\phi_1}{\partial s_1} & \dfrac{\partial\phi_1}{\partial s_2} \\ \dfrac{\partial\phi_2}{\partial s_1} & \dfrac{\partial\phi_2}{\partial s_2} \end{pmatrix}. \tag{65}$$

It can be proved that $\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O})$ is independent of the parameterization.
Further it can be proved that if $\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O}) = \mathbf{F}(\bar{\mathsf{p}}_\mathsf{A}) \quad \forall\bar{\mathsf{p}}_\mathsf{O}$, i.e. $\mathbf{F}$ is a uniform tensor field, then $\boldsymbol{\phi}$ is affine.

We assume that every deformation is such that

$$\det\mathbf{F}(\bar{\mathsf{p}}_\mathsf{O}) > 0 \quad \forall\bar{\mathsf{p}}_\mathsf{O}. \tag{66}$$

7 Displacement gradient

From the definition (6) of *displacement field* we get

$$\mathbf{u}(\bar{\mathsf{p}}_\mathsf{O}) = \boldsymbol{\phi}(\bar{\mathsf{p}}_\mathsf{O}) - \bar{\mathsf{p}}_\mathsf{O}, \tag{67}$$

which when evaluated along the curve (50) through $\bar{p}_O$ becomes

$$\mathbf{u}(\bar{c}(h)) = \boldsymbol{\phi}(\bar{c}(h)) - \bar{c}(h) = c(h) - \bar{c}(h), \tag{68}$$
$$\mathbf{u}(\bar{c}(0)) = \boldsymbol{\phi}(\bar{c}(0)) - \bar{c}(0) = c(0) - \bar{c}(0). \tag{69}$$

By using the definition of gradient of a vector field together with the definiton of deformation gradient, we obtain

$$\nabla\mathbf{u}(\bar{p}_O)\,\bar{c}' = c' - \bar{c}' = \mathbf{F}(\bar{p}_O)\bar{c}' - \bar{c}' = (\mathbf{F}(\bar{p}_O) - \mathbf{I})\bar{c}'. \tag{70}$$

Since this relation holds true for the derivative along any curve, then

$$\nabla\mathbf{u}(\bar{p}_O) = \mathbf{F}(\bar{p}_O) - \mathbf{I}. \tag{71}$$

A component description of the displacement field $\mathbf{u}$ can be given by defining on the parameterization domain two scalar functions u_1 e u_2 such that

$$\mathbf{u}(\boldsymbol{\kappa}(s_1, s_2)) = u_1(s_1, s_2)\mathbf{e}_1 + u_2(s_1, s_2)\mathbf{e}_2. \tag{72}$$

Since the displacement field on curves (40) and (45) is

$$\mathbf{u}(\bar{c}_1) = u_1(s_1 + h, s_2)\mathbf{e}_1 + u_2(s_1 + h, s_2)\mathbf{e}_2, \tag{73}$$
$$\mathbf{u}(\bar{c}_2) = u_1(s_1, s_2 + h)\mathbf{e}_1 + u_2(s_1, s_2 + h)\mathbf{e}_2, \tag{74}$$

from the defintion of gradient of a vector field we get

$$\nabla\mathbf{u}(\bar{p}_O)\,\bar{c}_1' = \partial_1 u_1\,\mathbf{e}_1 + \partial_1 u_2\,\mathbf{e}_2, \tag{75}$$
$$\nabla\mathbf{u}(\bar{p}_O)\,\bar{c}_2' = \partial_2 u_1\,\mathbf{e}_1 + \partial_2 u_2\,\mathbf{e}_2. \tag{76}$$

Hence the matrix of $\nabla\mathbf{u}(\bar{p}_O)$ turns out to be

$$[\nabla\mathbf{u}(\bar{p}_O)] = \begin{pmatrix} \dfrac{\partial u_1}{\partial s_1} & \dfrac{\partial u_1}{\partial s_2} \\ \dfrac{\partial u_2}{\partial s_1} & \dfrac{\partial u_2}{\partial s_2} \end{pmatrix}. \tag{77}$$

8 Local deformation

Note that if the deformation is affine then the line through $\bar{p}_O$ (39) is transformed into the line

$$c_1(h) = \boldsymbol{\phi}(\bar{p}_O) + h\,\mathbf{F}(\bar{p}_O)\mathbf{e}_1. \tag{78}$$

If the deformation is not affine then (78) is not true in general and we can define the difference

$$\mathfrak{o}(h) := c_1(h) - \big(\boldsymbol{\phi}(\bar{p}_O) + h\,\mathbf{F}(\bar{p}_O)\mathbf{e}_1\big) = \big(c_1(h) - c_1(0)\big) - h\,\mathbf{F}(\bar{p}_O)\mathbf{e}_1, \tag{79}$$

which, since by (60)

$$\lim_{h\to 0}\frac{1}{h}\big(c_1(h) - c_1(0)\big) = c_1{}' = \mathbf{F}(\bar{p}_O)\mathbf{e}_1, \tag{80}$$

has the property

$$\lim_{h\to 0}\frac{\|\mathfrak{o}(h)\|}{|h|} = 0. \tag{81}$$

Hence for a generic deformation, by using the definition (79), the expression (78) is replaced by

$$c_1(h) = \boldsymbol{\phi}(\bar{p}_O) + h\,\mathbf{F}(\bar{p}_O)\mathbf{e}_1 + \mathfrak{o}(h). \tag{82}$$

Note that, because of (81), $\|\mathfrak{o}(h)\|$ goes to zero faster than h: we can say that sufficiently close to $\bar{p}_O$ every deformation $\boldsymbol{\phi}$ is an affine deformation.

8.1 Stretch and shear

The stretch $\mathbf{U}$ describes how lines through any position are transformed by an affine deformation: they are stretched while the angles between them are changed. Difference vectors between any two positions along the principal directions are stretched, while keeping their direction, by a factor equal to the corresponding eigenvalue of $\mathbf{U}$.

In general, a non affine deformation transforms lines into curves, but the stretch $\mathbf{U}(\bar{\mathsf{p}}_O)$, given by the polar decomposition of $\mathbf{F}(\bar{\mathsf{p}}_O)$, still describes how tangent vectors to curves through $\bar{\mathsf{p}}_O$ are transformed: they are stretched while the angles between them are changed. Tangent vectors which are eigenvectors of $\mathbf{U}(\bar{\mathsf{p}}_O)$ are stretched, while keeping their direction, by a factor equal to the corresponding eigenvalue of $\mathbf{U}$.

Since $\mathbf{F}(\bar{\mathsf{p}}_O)$ depends on the position $\bar{\mathsf{p}}_O$ also the eigenvalues and the eigenvectors of $\mathbf{U}$ depend on $\bar{\mathsf{p}}_O$.

For any position $\bar{\mathsf{p}}_O$ and any vector $\mathbf{a}$ we call *stretch* and *elongation* along the direction $\mathbf{a}$, respectively

$$\lambda := \frac{\|\mathbf{U}\mathbf{a}\|}{\|\mathbf{a}\|}, \qquad \varepsilon := \frac{\|\mathbf{U}\mathbf{a}\| - \|\mathbf{a}\|}{\|\mathbf{a}\|} = \lambda - 1. \tag{83}$$

Along each principal direction, since $\mathbf{U}\mathbf{a}_i = \lambda_i \mathbf{a}_i$ the stretch is $\lambda = \lambda_i$. The stretch along a principal directions is called *principal stretch*.

Two orthogonal tangent vectors $\mathbf{a}_1$ e $\mathbf{a}_2$ at $\bar{\mathsf{p}}_O$ are transformed by $\mathbf{U}(\bar{\mathsf{p}}_O)$ into linearly independent vectors which are not orthogonal in general. We call *shear strain* between $\mathbf{a}_1$ e $\mathbf{a}_2$ the angle $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ such that

$$\cos\left(\frac{\pi}{2} - \gamma\right) = \frac{\mathbf{U}\mathbf{a}_1 \cdot \mathbf{U}\mathbf{a}_2}{\|\mathbf{U}\mathbf{a}_1\|\|\mathbf{U}\mathbf{a}_2\|}. \tag{84}$$

Note that if $\mathbf{a}_1$ e $\mathbf{a}_2$ eigenvectors of $\mathbf{U}$ then $\gamma = 0$.