Elastic affine bodies

Contents

1 Balance principles

1.1 Rigid body

A *rigid body* is a body whose placements are such that the deformation between any two of them is rigid. The space of test velocity fields is made of all the rigid velocity fields.

Let us assume the following principle: *in any motion at each time t the forces are such that their power in any test velocity field is zero*:

$$\mathcal{W}^{(out)}(\mathbf{v}) = 0 \quad \forall \mathbf{v}. \tag{1}$$

Since for any rigid test velocity field the power takes the form

$$\mathcal{W}^{(out)}(\mathbf{v}) = \mathbf{f} \cdot \mathbf{v}_{\mathsf{O}} + \mathbf{M}_{\mathsf{p_O}} \cdot \mathbf{W}, \tag{2}$$

where $\mathbf{f}$ is the total force and $\mathbf{M}_{\mathsf{p_O}}$ is the total moment with respect to any position $\mathsf{p_O}$, the principle above is equivalent to the following *balance equations*

$$\mathbf{f} = \mathbf{o}, \tag{3}$$
$$\operatorname{skw} \mathbf{M}_{\mathsf{p_O}} = \mathbf{O}. \tag{4}$$

1.2 Affine body and Cauchy stress

An *affine body* is a body whose placements are such that the deformation between any two of them is affine. The space of the test velocity fields is made of all the affine velocity fields.

The affine body is the simplest model of *deformable* body. Since for an affine test velocity the power is

$$\mathcal{W}^{(out)}(\mathbf{v}) = \mathbf{f} \cdot \mathbf{v}_{\mathsf{O}} + \mathbf{M}_{\mathsf{p_O}} \cdot \mathbf{L}, \tag{5}$$

if we assumed that at any time t the power be zero for any test velocity field, we would get the following balance equations

$$\mathbf{f} = \mathbf{o}, \tag{6}$$
$$\operatorname{skw} \mathbf{M}_{\mathsf{p_O}} = \mathbf{O}, \tag{7}$$
$$\operatorname{sym} \mathbf{M}_{\mathsf{p_O}} = \mathbf{O}. \tag{8}$$

The last condition will filter out any force distribution with a symmetric moment tensor. In order to allow for such force distributions we can introduce an *inner power*

$$\mathcal{W}^{(in)}(\mathbf{v}) = -(\mathbf{z} \cdot \mathbf{v}_{\mathsf{O}} + \mathbf{T} \cdot \mathbf{L})\, V_{\mathcal{R}}, \tag{9}$$

and assume that *in any motion at each time t*

$$\mathcal{W}^{(out)}(\mathbf{v}) + \mathcal{W}^{(in)}(\mathbf{v}) = 0 \tag{10}$$

for any affine test velocity field. Such a new *balance principle* turns out to be equivalent to the *balance equations*

$$\mathbf{f} - \mathbf{z}\, V_{\mathcal{R}} = \mathbf{o}, \tag{11}$$
$$\mathbf{M}_{\mathsf{p_O}} - \mathbf{T}\, V_{\mathcal{R}} = \mathbf{O}. \tag{12}$$

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2 Stress characterization

Since the *inner power* $\mathcal{W}^{(in)}$ has been introduced for a body which can undergo non rigid deformations, it is reasonable to assume that it vanishes for any rigid test velocity field. Hence

$$\mathbf{z} \cdot \mathbf{v}_{\mathsf{O}} + \mathbf{T} \cdot \mathbf{W} = 0, \tag{13}$$

for any skew symmetric tensor $\mathbf{W}$. This condition is equivalent to

$$\mathbf{z} = \mathbf{o}, \tag{14}$$
$$\operatorname{skw} \mathbf{T} = \mathbf{O}. \tag{15}$$

The *balance equations* for an affine body become

$$\mathbf{f} = \mathbf{o}, \tag{16}$$
$$\operatorname{skw} \mathbf{M} = \mathbf{O}, \tag{17}$$
$$\operatorname{sym} \mathbf{M} = \mathbf{T}\, V_{\mathcal{R}}. \tag{18}$$

It is worth noting that the moment tensor turns out to be independent of $\mathsf{p_O}$ because the total force is zero.

2.1 Objectivity

In a more general setting we consider two different "observers". An affine motion is seen by the first observer as described by $\boldsymbol{\phi}$, such that

$$\boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{A}}, t) = \boldsymbol{\phi}(\bar{\mathsf{p}}_{\mathsf{O}}, t) + \mathbf{F}(t)(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}) \qquad \forall \mathsf{A} \in \mathcal{B} \tag{19}$$

which can be rewritten as

$$\mathsf{p}_{\mathsf{A}}(t) = \mathsf{p}_{\mathsf{O}}(t) + \mathbf{F}(t)(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}). \tag{20}$$

A different observer will see the same body point A at time t in a different position given by

$$\mathsf{p}^*_{\mathsf{A}}(t) = \mathsf{q}^*(t) + \mathbf{Q}(t)(\mathsf{p}_{\mathsf{A}}(t) - \mathsf{q}(t)), \tag{21}$$

where $\mathbf{Q}$, q, q^* are three time functions, and $\mathbf{Q}(t)$ is an orthogonal tensor. Since

$$\mathsf{p}^*_{\mathsf{O}}(t) = \mathsf{q}^*(t) + \mathbf{Q}(t)(\mathsf{p}_{\mathsf{O}}(t) - \mathsf{q}(t)), \tag{22}$$

subtracting (22) from (21) we get, using (20) and dropping the argument t,

$$\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}} = \mathbf{Q}(\mathsf{p}_{\mathsf{A}} - \mathsf{p}_{\mathsf{O}}) = \mathbf{QF}(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}). \tag{23}$$

Hence

$$\mathsf{p}^*_{\mathsf{A}} = \mathsf{p}^*_{\mathsf{O}} + \mathbf{F}^*(\bar{\mathsf{p}}_{\mathsf{A}} - \bar{\mathsf{p}}_{\mathsf{O}}) \tag{24}$$

with

$$\mathbf{F}^* = \mathbf{QF}. \tag{25}$$

By differentiating (21) with respect to time, we get the relation between the velocities measured by the two observers

$$\begin{aligned} \dot{\mathsf{p}}^*_{\mathsf{A}} &= \dot{\mathsf{q}}^* + \dot{\mathbf{Q}}(\mathsf{p}_{\mathsf{A}} - \mathsf{q}) + \mathbf{Q}(\dot{\mathsf{p}}_{\mathsf{A}} - \dot{\mathsf{q}}) \\ &= \dot{\mathsf{q}}^* + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{q}^*) + \mathbf{Q}(\dot{\mathsf{p}}_{\mathsf{A}} - \dot{\mathsf{q}}) \\ &= \mathbf{Q}\dot{\mathsf{p}}_{\mathsf{A}} + (\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{q}^*). \end{aligned} \tag{26}$$

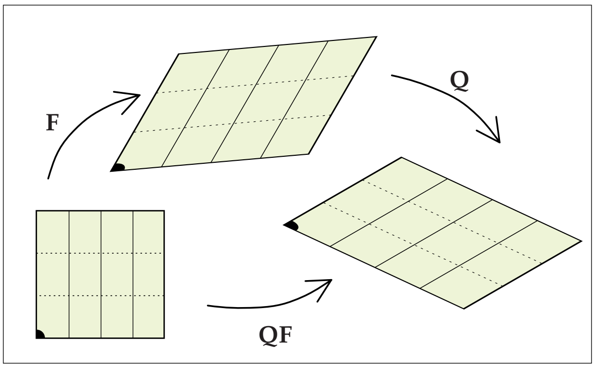

Figure 1: Change of observer

For any given test velocity field

$$\mathbf{v}_{\mathsf{A}} = \mathbf{v}_{\mathsf{O}} + \mathbf{L}(\mathsf{p}_{\mathsf{A}} - \mathsf{p}_{\mathsf{O}}), \tag{27}$$

in a change of observer the *corresponding test velocity field* is the field

$$\mathbf{v}^*_{\mathsf{A}} = \mathbf{v}^*_{\mathsf{O}} + \mathbf{L}^*(\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}}) \tag{28}$$

such that

$$\mathbf{v}^*_{\mathsf{A}} = \mathbf{Q}\mathbf{v}_{\mathsf{A}} + (\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{q}^*). \tag{29}$$

Replacing A with O we get

$$\mathbf{v}^*_{\mathsf{O}} = \mathbf{Q}\mathbf{v}_{\mathsf{O}} + (\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{O}} - \mathsf{q}^*), \tag{30}$$

and, taking the difference,

$$\begin{aligned}
\mathbf{v}^*_{\mathsf{A}} - \mathbf{v}^*_{\mathsf{O}} &= \mathbf{Q}(\mathbf{v}_{\mathsf{A}} - \mathbf{v}_{\mathsf{O}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}})\\
&= \mathbf{Q}\mathbf{L}(\mathsf{p}_{\mathsf{A}} - \mathsf{p}_{\mathsf{O}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}})\\
&= \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}})\\
&= (\mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathsf{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}})(\mathsf{p}^*_{\mathsf{A}} - \mathsf{p}^*_{\mathsf{O}}).
\end{aligned} \tag{31}$$

Thus

$$\mathbf{L}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathsf{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}. \tag{32}$$

The *material objectivity principle* is stated as follows: *the inner power, for any test velocity field, is invariant under a change of observer*.

As a consequence we get that at any time t and for any change of observer, as defined by the three time functions $\mathbf{Q}$, q, q^*,

$$\mathbf{z}^* \cdot \mathbf{v}^*_{\mathsf{O}} + \mathbf{T}^* \cdot \mathbf{L}^* = \mathbf{z} \cdot \mathbf{v}_{\mathsf{O}} + \mathbf{T} \cdot \mathbf{L} \tag{33}$$

for any test velocity field. By using (30) and (32) we obtain

$$\begin{aligned}
&\mathbf{z}^* \cdot \Big(\mathbf{Q}\mathbf{v}_{\mathsf{O}} + (\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{O}} - \mathsf{q}^*)\Big) - \mathbf{z} \cdot \mathbf{v}_{\mathsf{O}}\\
&\quad + \mathbf{T}^* \cdot \Big(\mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathsf{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}\Big) - \mathbf{T} \cdot \mathbf{L} = 0.
\end{aligned} \tag{34}$$

Selecting only those changes of observer such that $\dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}} = \mathbf{O}$, $\dot{\mathsf{q}}^* = \mathbf{Q}\dot{\mathsf{q}}$ we get

$$\left(\mathbf{Q}^{\mathsf{T}}\mathbf{z}^* - \mathbf{z}\right) \cdot \mathbf{v}_{\mathsf{O}} + \left(\mathbf{Q}^{\mathsf{T}}\mathbf{T}^*\mathbf{Q} - \mathbf{T}\right) \cdot \mathbf{L} = 0 \tag{35}$$

which must hold for any test velocity field. Hence

$$\mathbf{z} = \mathbf{Q}^{\mathsf{T}}\mathbf{z}^*, \tag{36}$$
$$\mathbf{T} = \mathbf{Q}^{\mathsf{T}}\mathbf{T}^*\mathbf{Q}. \tag{37}$$

Replacing these expressions into (34) we obtain

$$\mathbf{z}^* \cdot \left((\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}}) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}(\mathsf{p}^*_{\mathsf{O}} - \mathsf{q}^*)\right) + \mathbf{T}^* \cdot \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}} = 0. \tag{38}$$

Selecting those changes of observer such that $\dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}} = \mathbf{O}$ we get

$$\mathbf{z}^* \cdot (\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}}) = 0 \tag{39}$$

which must hold whatever be the vector $(\dot{\mathsf{q}}^* - \mathbf{Q}\dot{\mathsf{q}})$. Hence

$$\mathbf{z}^* = \mathbf{o}, \tag{40}$$

while (38) becomes

$$\mathbf{T}^* \cdot \dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}} = 0. \tag{41}$$

In order for this condition to hold for any change of observer the stress must be such that

$$\operatorname{skw} \mathbf{T}^* = \mathbf{O}\,, \tag{42}$$

since $\dot{\mathbf{Q}}\mathbf{Q}^{\mathsf{T}}$ is skew symmetric. It is worth noting that (40) and (42) implies again (14) and (15). For from (36) and (40) we get

$$\mathbf{z} = \mathbf{Q}^{\mathsf{T}}\mathbf{z}^* = \mathbf{o}\,, \tag{43}$$

while from (37) and (42) we get

$$\begin{aligned}\operatorname{skw} \mathbf{T} &= \frac{1}{2}\left(\mathbf{Q}^{\mathsf{T}}\mathbf{T}^*\mathbf{Q} - \mathbf{Q}^{\mathsf{T}}(\mathbf{T}^*)^{\mathsf{T}}\mathbf{Q}\right) = \mathbf{Q}^{\mathsf{T}}\frac{1}{2}\left(\mathbf{T}^* - (\mathbf{T}^*)^{\mathsf{T}}\right)\mathbf{Q}\\ &= \mathbf{Q}^{\mathsf{T}}\,(\operatorname{skw} \mathbf{T}^*)\,\mathbf{Q} = \mathbf{O}\,.\end{aligned} \tag{44}$$

2.2 Composition with a rigid motion

A different point of view, which turns out to be equivalent to the previous one, consists in composing the affine motion (19) with a rigid motion as described, at time t, by the expression

$$\boldsymbol{\phi}_r(\mathsf{p}_{\mathsf{A}}(t), t) = \boldsymbol{\phi}_r(\mathsf{q}(t), t) + \mathbf{Q}(t)(\mathsf{p}_{\mathsf{A}}(t) - \mathsf{q}(t)). \tag{45}$$

If we set

$$\mathsf{p}^*_{\mathsf{A}}(t) = \boldsymbol{\phi}_r(\mathsf{p}_{\mathsf{A}}(t), t), \tag{46}$$
$$\mathsf{q}^*(t) = \boldsymbol{\phi}_r(\mathsf{q}(t), t), \tag{47}$$

then (45) turns into (21). Restating the *material objectivity principle* as follows: *the inner power, for any test velocity field be invariant for any composition with a rigid motion*, we get at the same conclusions we arrived at by stating the principle in terms of invariance under a change of observer.

3 Material response

The characterization of the relationship between stress and motion is based on these principles:

- Principle of determinism: *the stress is determined by the past history of the motion*.
- Principle of local action: *the stress in a body point does not depend on the deformation in any other point at a finite distance*.

Let us consider an affine motion as described by the expression (19). At any time the deformation is defined by its value in p_O and by the deformation gradient $\mathbf{F}$. Since by the principle of local action the stress cannot depend on the value of the deformation in a particular position, then it will depend on the gradient. Hence the *response function* at time t gets the following form

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}^t), \tag{48}$$

where $\mathbf{F}^t$ denotes the history of the motion through the deformation gradient. A very important class of materials is made up of the *elastic materials*, characterized by the following property: *the stress depends only on the current value of the deformation gradient*. The response function becomes

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}). \tag{49}$$

Note that the response function contains the description of a placement $\bar{p}$, on which the deformation $\boldsymbol{\phi}$ applies, characterized by a zero stress.

Let us consider for a body made up of an elastic material the affine motion (20) and the corresponding motion (24) as seen by a different observer through (21). The response from the point of view of the second observer will be, by (25),

$$\mathbf{T}^* = \widehat{\mathbf{T}}(\mathbf{F}^*) = \widehat{\mathbf{T}}(\mathbf{QF}). \tag{50}$$

Since the *principle of material objectivity* leads to (37), by replacing there (49) and (50) we get

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{Q}^{\mathsf{T}}\widehat{\mathbf{T}}(\mathbf{QF})\mathbf{Q} \tag{51}$$

This condition has to be fulfilled by the response function for any $\mathbf{Q}$. By choosing $\mathbf{Q} = \mathbf{R}^{\mathsf{T}}$, where $\mathbf{R}$ is the rotation in

$$\mathbf{F} = \mathbf{RU}, \tag{52}$$

we get as a necessary condition to be fulfilled

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^{\mathsf{T}}, \tag{53}$$

or equivalently

$$\mathbf{R}^{\mathsf{T}}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{R} = \widehat{\mathbf{T}}(\mathbf{U}). \tag{54}$$

Viceversa, if a response function has the property (53), then it satisfies (37) for any $\mathbf{Q}$. For, replacing (53) in (50) we get

$$\mathbf{T}^* = \widehat{\mathbf{T}}(\mathbf{QF}) = (\mathbf{QR})\widehat{\mathbf{T}}(\mathbf{U})(\mathbf{QR})^{\mathsf{T}} = \mathbf{Q}(\mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^{\mathsf{T}})\mathbf{Q}^{\mathsf{T}} = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^{\mathsf{T}} = \mathbf{QTQ}^{\mathsf{T}} \tag{55}$$

which is equivalent to (37). Hence the property (53) characterizes all the elastic materials and is called *reduced form of the response function for elastic materials*.

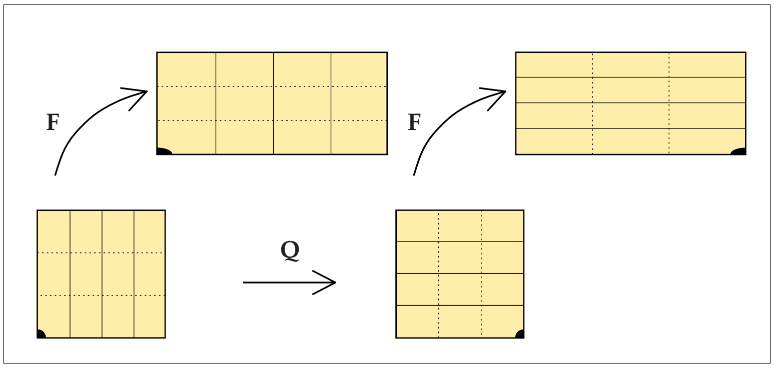

Figure 2: A rotation $\mathbf{Q}$ belongs to the material symmetry group if $\widehat{\mathbf{T}}(\mathbf{FQ}) = \widehat{\mathbf{T}}(\mathbf{F})$.

4 Material symmetry group

Note that if the affine deformation $\boldsymbol{\phi}$ is applied *after* a rigid deformation $\boldsymbol{\phi}_r$ such that

$$\bar{\mathsf{p}}_A = \boldsymbol{\phi}_r(\check{\mathsf{p}}_A) = \boldsymbol{\phi}_r(\check{\mathsf{p}}_O) + \mathbf{Q}(\check{\mathsf{p}}_A - \check{\mathsf{p}}_O) \tag{56}$$

then the composition $\boldsymbol{\phi}^\star := \boldsymbol{\phi} \circ \boldsymbol{\phi}_r$ is again an affine deformation such that

$$\boldsymbol{\phi}^\star(\check{\mathsf{p}}_A) = \boldsymbol{\phi}(\boldsymbol{\phi}_r(\check{\mathsf{p}}_A)) = \boldsymbol{\phi}(\boldsymbol{\phi}_r(\check{\mathsf{p}}_O)) + \mathbf{FQ}(\check{\mathsf{p}}_A - \check{\mathsf{p}}_O). \tag{57}$$

Since the deformation gradient turns out to be $\mathbf{F}^\star = \mathbf{FQ}$, the corresponding stress is

$$\mathbf{T}^\star = \widehat{\mathbf{T}}(\mathbf{F}^\star) = \widehat{\mathbf{T}}(\mathbf{FQ}). \tag{58}$$

The group made up of the rotations $\mathbf{Q}$ such that, when followed by the same deformation, leave the response unchanged, i.e. such that

$$\widehat{\mathbf{T}}(\mathbf{FQ}) = \widehat{\mathbf{T}}(\mathbf{F}) \quad \forall \mathbf{F}, \tag{59}$$

is called *symmetry group of the material*.

As an example, if the rotation $\mathbf{Q}$ in Fig 2 and in Fig. 3 belongs to the symmetry group, any $\mathbf{F}$ gives rise to the same stress. It is worth noting that the application of $\mathbf{F}$ after $\mathbf{Q}$ leads to a different configuration than the one reached by applying just $\mathbf{F}$, even though the shapes are the same.

4.1 Isotropy

Those materials whose symmetry group is the whole rotation group of $\mathcal{V}$ are called *isotropic*. Since for isotropic materials (59) holds for any rotation $\mathbf{Q}$ it should hold in particular for $\mathbf{Q} = \mathbf{R}^\mathsf{T}$, thus becoming

$$\widehat{\mathbf{T}}(\mathbf{RUR}^\mathsf{T}) = \widehat{\mathbf{T}}(\mathbf{F}), \tag{60}$$

which on turn, by (53), becomes

$$\widehat{\mathbf{T}}(\mathbf{RUR}^\mathsf{T}) = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^\mathsf{T}. \tag{61}$$

Viceversa, any material whose response function satisfies (61) turns out to be isotropic. For, because the polar decomposition of $\mathbf{FQ}$ is

$$\mathbf{FQ} = \mathbf{RUQ} = (\mathbf{RQ})(\mathbf{Q}^\mathsf{T}\mathbf{UQ}), \tag{62}$$

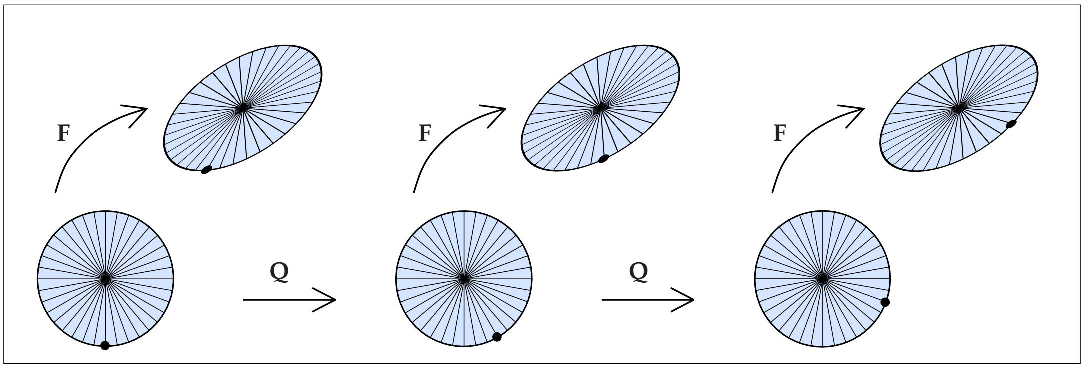

Figure 3: A material is isotropic if $\widehat{\mathsf{T}}(\mathbf{FQ}) = \widehat{\mathsf{T}}(\mathbf{F})$ for any rotation $\mathbf{Q}$.

by applying (53) first, then (61) and again (53), we get

$$\widehat{\mathsf{T}}(\mathbf{FQ}) = (\mathbf{RQ})\widehat{\mathsf{T}}(\mathbf{Q}^\mathsf{T}\mathbf{UQ})(\mathbf{RQ})^\mathsf{T} = \mathbf{R}\widehat{\mathsf{T}}(\mathbf{U})\mathbf{R}^\mathsf{T} = \widehat{\mathsf{T}}(\mathbf{F}) \tag{63}$$

5 Piola-Kirchhoff stress

The expression for the inner power can be written replacing the current volume with the volume before deformation

$$\mathbf{T} \cdot \mathbf{L}\, V_{\mathcal{R}} = \mathbf{T} \cdot \mathbf{L} \det \mathbf{F}\, V_{\bar{\mathcal{R}}}. \tag{64}$$

This allows us to define a stress $\mathbf{S}$, depending on $\mathbf{T}$, in the following way. In an affine test velocity field the difference between the velocities at p_A and p_O è given by

$$\mathbf{L}(\mathsf{p}_A - \mathsf{p}_O). \tag{65}$$

Since

$$\mathbf{L}(\mathsf{p}_A - \mathsf{p}_O) = \mathbf{LF}(\bar{\mathsf{p}}_A - \bar{\mathsf{p}}_O), \tag{66}$$

the same difference can be obtained by applying $\mathbf{LF}$ to the difference between the corresponding positions in $\bar{\mathcal{R}}$. We can define a new stress as the tensor $\mathbf{S}$ such that

$$\mathbf{T} \cdot \mathbf{L}\, V_{\mathcal{R}} = \mathbf{S} \cdot \mathbf{LF}\, V_{\bar{\mathcal{R}}}, \qquad \forall \mathbf{L}. \tag{67}$$

s By (64) we get

$$\mathbf{T} \cdot \mathbf{L} \det \mathbf{F}\, V_{\bar{\mathcal{R}}} = \mathbf{S} \cdot \mathbf{LF}\, V_{\bar{\mathcal{R}}}, \tag{68}$$

and then

$$\mathbf{T} \cdot \mathbf{L} \det \mathbf{F} = \mathbf{SF}^\mathsf{T} \cdot \mathbf{L}, \qquad \forall \mathbf{L}. \tag{69}$$

Hence it turns out that

$$\mathbf{T} \det \mathbf{F} = \mathbf{SF}^\mathsf{T}, \tag{70}$$

from which we get the sought expression

$$\mathbf{S} = \mathbf{T}(\mathbf{F}^\mathsf{T})^{-1} \det \mathbf{F}. \tag{71}$$

6 Constraints and reactive forces

Let us consider an affine body whose points, possibly only a subset of them, are *constrained* to move on a surface or along a curve or simply to stay still. From our every day experience we know that we can apply a load over a body laying on a table, though those forces do not satisfy the balance equations (16) and (17). That is why we have to admit that other forces do exist, which we call *reactive*, that are different from the forces we can assign as a constant or as a function of time or through a more general *constitutive* function depending on the motion. Further, it is reasonable to assume that reactive forces are orthogonal to any trajectory allowed by the constraints.

Hence we assume that the *outer* power is the sum of the power of the *assigned* forces and the power of the *reactive* forces

$$\mathcal{W}^{(out)}(\mathbf{v}) = (\mathbf{f}^a + \mathbf{f}^r) \cdot \mathbf{v}_{\mathsf{O}} + (\mathbf{M}^a_{\mathsf{p_O}} + \mathbf{M}^r_{\mathsf{p_O}}) \cdot \mathbf{L}. \tag{72}$$

Denoting by $\mathbf{v}^{\mathrm{v}}_{\mathsf{O}}$ and $\mathbf{L}^{\mathrm{v}}$ the descriptors of the affine test velocity fields *compatible with the constraints*, we introduce the following principle

$$\mathbf{f}^r \cdot \mathbf{v}^{\mathrm{v}}_{\mathsf{O}} + \mathbf{M}^r_{\mathsf{p_O}} \cdot \mathbf{L}^{\mathrm{v}} = 0, \quad \forall \mathbf{v}^{\mathrm{v}}_{\mathsf{O}}, \ \forall \mathbf{L}^{\mathrm{v}} \tag{73}$$

which reads: *the power of the reactive forces in any test velocity field compatible with the constraints is zero*.